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TWISTED $N = 2$ SUPERSYMMETRY WITH CENTRAL CHARGE AND EQUIVARIANT COHOMOLOGY

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ABSTRACT

We present an equivariant extension of the Thom form with respect to a vector field action, in the framework of the Mathai-Quillen formalism. The associated Topological Quantum Field Theories correspond to twisted $N = 2$ supersymmetric theories with a central charge. We analyze in detail two different cases: topological sigma models and non-abelian monopoles on four-manifolds.

1. Introduction

It is by now a well-known fact that many $N = 2$ supersymmetric theories can be reformulated through a “twisting” of the supersymmetry algebra in order to construct Topological Quantum Field Theories. The classical examples of this procedure are the Donaldson-Witten theory [1] and the topological sigma model [2], which arise by twisting the $N = 2$ supersymmetric Yang-Mills theory and the $N = 2$ supersymmetric sigma model, respectively. The most useful approach to understand the geometry involved in this kind of models is perhaps the one based on the Mathai-Quillen formalism [3]. It was shown in [4] that the topological lagrangian appearing in Donaldson-Witten theory can be considered as the Euler class of a certain infinite-dimensional bundle over the space of Yang-Mills connections. The Euler class is obtained as the pullback of the Thom class of the bundle by means of a section whose zero locus is precisely the moduli space of anti-self-dual instantons of Donaldson theory [5, 6, 7]. The representative of the Thom class that appears in Donaldson-Witten theory is precisely the one appearing in [3]. Subsequently it was shown that the same construction holds in the case of topological sigma models [8, 9]. A review of these developments can be found in [10, 11]. In the same way, one can use the Mathai-Quillen formalism to construct Topological Quantum Field Theories starting from a moduli problem formulated in a purely geometrical setting.

However there are some twisted $N = 2$ supersymmetric theories which do not have a clear formulation in the Mathai-Quillen framework, and therefore their geometrical structure is not very well understood. One should then look for generalizations of this formalism to take into account the rich topological structures hidden in the supersymmetry algebra. The purpose of this paper is precisely to obtain an equivariant extension of the Thom class of a bundle with respect to a vector field action, in the Mathai-Quillen setting. This construction can be regarded as a generalization of the equivariant extensions of the curvature considered in [12, 13, 14, 15]. Apart from its mathematical interest, it turns out that the Topologi-

cal Quantum Field Theories constructed with this extension correspond to twisted $N = 2$ supersymmetric theories with a central charge. We will consider in detail two different applications of our construction. The first one will be a topological sigma model with a vector field action on the target space. The resulting theory corresponds to the twisted $N = 2$ supersymmetric sigma model with potentials constructed in [16]. Our second example will be non-abelian monopoles on four manifolds [17], where the vector field action is now given by a $U(1)$ symmetry acting on the monopole fields. The topological lagrangian that one obtains in this way can be regarded as a topological Yang-Mills theory coupled to twisted massive hypermultiplets. This twisted model was also considered in [18], where the relation to equivariant cohomology was pointed out.

These two examples are very interesting from the topological point of view. The first one gives the natural framework to consider equivariant quantum cohomology of almost-hermitean manifolds with a vector field action. The four-dimensional example gives a very explicit connection between $N = 2$ quantum field theories and the strategy proposed by Pidstrigach and Tyurin [19] to prove the equivalence between Donaldson and Seiberg-Witten invariants using non-abelian monopole equations.

The organization of this paper is as follows. In section 2 we review some results on equivariant cohomology and on the construction of equivariant extensions of the curvature. In section 3 we present the equivariant extension of the Thom form in the Mathai-Quillen formalism. We consider different geometrical situations which roughly correspond to the Weil or Cartan representatives of the usual Mathai-Quillen form. In section 4 we apply the previous results to topological sigma models and non-abelian monopoles on four manifolds, from a purely geometrical point of view. In section 5 we consider the twisting of $N = 2$ supersymmetry with a central charge and we relate it to the equivariant cohomology associated to a vector field action. We also rederive the two models of section 4 by twisting the $N = 2$ supersymmetric sigma model with potentials and the $N = 2$ supersymmetric Yang-Mills theory coupled to massive matter hypermultiplets. Finally, in section

6 we state our final remarks and conclusions, and some prospects for future work.

2. Equivariant cohomology and equivariant curvature

2.1 EQUIVARIANT COHOMOLOGY

In this paper we will use the Cartan model for equivariant cohomology, and here we will review some basic definitions. For a detailed account of equivariant cohomology, see [12, 3,10].

Let X be a vector field acting on a manifold M . Recall that every vector field is associated to a locally defined one-parameter group of transformations of M , $\phi : I \times M \rightarrow M$, with $I \subset \mathbb{R}$ being an open interval containing $t = 0$. If we put $\phi_m(t) = \phi_t(m) = \phi(t, m)$, the vector field corresponding to ϕ is given by:

$$X(m) = \phi_{m*0} \left(\frac{d}{dt} \right)_{t=0}, \quad (2.1)$$

where $*$ denotes as usual the differential map between tangent spaces. A particular case of this correspondence is a circle ($U(1)$) action on M with generator X .

Let $\mathcal{L}(X)$ be the Lie derivative with respect to the vector field X , and let $\Omega^*(M)$ be the complex of differential forms on M . We denote by $\Omega_X^*(M)$ the kernel of $\mathcal{L}(X)$ in $\Omega^*(M)$. We consider now the polynomial ring generated by a generator u of degree 2 over $\Omega^*(M)$, denoted by $\Omega^*(M)[u]$. On this ring we define the equivariant exterior derivative as follows:

$$d_X \omega = d\omega - u(X)\omega, \quad \omega \in \Omega^*[u], \quad (2.2)$$

where $\iota(X)$ denotes the usual inner product with the vector field X . Notice that

$$d_X^2 = -u\mathcal{L}(X), \quad (2.3)$$

and therefore d_X is nilpotent on $\Omega_X^*(M)[u]$. Elements of $\Omega_X^*[u]$ are called *equivariant differential forms*. An equivariant differential form ω verifying $d_X \omega = 0$ is said to be *equivariantly closed*. Notice that, if $\omega \in \Omega^*(M)[u]$ and $d_X \omega = 0$, necessarily $\omega \in \Omega_X^*(M)[u]$ because of (2.3).

Given a closed invariant differential form, *i.e.*, a form $\omega \in \Omega_X^*$ with $d\omega = 0$, we don't get an equivariantly closed differential form unless $\iota(X)\omega = 0$. But it might be possible to find some polynomial p in the ideal generated by u in $\Omega_X^*(M)[u]$ such that the resulting form $\omega' = \omega + p$ is equivariantly closed. The form ω' is called an *equivariant extension* of ω . One of the purposes of this paper is to find an equivariant extension of the Thom class of a vector bundle under suitable conditions, in the framework of the Mathai-Quillen formalism. As the Mathai-Quillen form involves the curvature of the vector bundle, we need an explicit expression for the equivariant extension of the curvature form. This has been done by Atiyah and Bott [12] following previous results by Bott in [13, 14], and by Berline and Vergne in [15]. Here we will review this construction for general vector bundles from the point of view of equivariant cohomology, and we will proceed in the same way to obtain the equivariant extension of the curvature for principal bundles [15]. Both results will be needed in the forthcoming subsections.

2.2 EQUIVARIANT CURVATURE FOR VECTOR BUNDLES

Let $\pi : E \rightarrow M$ be a real vector bundle. We suppose that there is a vector field X acting on M , and also an "action" of this field on E compatible with the action on M . With this we mean [12, 14] that there is a differential operator Λ acting on the space of sections of E , $\Gamma(E)$:

$$\Lambda : \Gamma(E) \rightarrow \Gamma(E), \quad (2.4)$$

that satisfies the derivation property

$$\Lambda(fs) = (Xf)s + f\Lambda s, \quad f \in C^\infty(M), \quad s \in \Gamma(E). \quad (2.5)$$

We will be particularly interested in the case in which there is a vector field X_E acting on E in a compatible way with the action of X on M . With this we mean the following: let $\hat{\phi}_t, \phi_t$ be the one-parameter flows corresponding to X_E, X , respectively. Then the following conditions are verified:

i) $\pi\hat{\phi}_t = \phi_t\pi$, i.e., the one-parameter flows intertwine with the projection map of the bundle.

ii) the map $E_m \rightarrow E_{\phi_t m}$ between fibres is a vector space homomorphism.

Notice that, if X_E , X are associated to circle actions on E , M , the above conditions simply state that E is a G -bundle over the G -space M , with $G = U(1)$. An obvious consequence of (i) is that X_E and X are π -related:

$$\pi_* X_E = X. \quad (2.6)$$

When there is a vector field X_E acting on E in the above way the operator Λ is naturally defined as:

$$(\Lambda s)(m) = \lim_{t \rightarrow 0} \frac{1}{t} [s(m) - \hat{\phi}_t s(\phi_{-t}(m))]. \quad (2.7)$$

It's easy to see that, because of condition (i) above, $\hat{\phi}_t s(\phi_{-t}(m))$ is in fact a section of E , and using (ii) one can check that the derivation property (2.5) holds. We say that the section $s \in \Gamma(E)$ is *invariant* if $\hat{\phi}_t s(\phi_{-t}(m)) = s(m)$, for all $t \in I$, $m \in M$. This is equivalent to $\Lambda s = 0$. If s is an invariant section, X_E and X are also s -related:

$$s_* X = X_E. \quad (2.8)$$

Consider now a connection D on the real vector bundle E of rank q . We say that D is *equivariant* if it conmmutes with the operator Λ . Let's write this condition with respect to a frame field $\{s_i\}_{i=1, \dots, q}$ on an open set $U \subset M$. We define the matrix-valued function and one-form on U , Λ_i^j , θ_i^j , by:

$$\Lambda s_i = \Lambda_i^j s_j, \quad D s_i = \theta_i^j s_j. \quad (2.9)$$

Of course, θ_i^j is the usual connection matrix. Under a change of frame $s' = sg$, where $g \in Gl(q, \mathbf{R})$, we can use the derivation property of Λ to obtain the matrix

with respect to the new local frame:

$$\Lambda' = g^{-1}\Lambda g + g^{-1}Xg. \quad (2.10)$$

Imposing $\Lambda D = D\Lambda$ on the local frame $\{s_i\}$ one gets

$$d\Lambda_i^j + \theta_k^j \Lambda_i^k = \mathcal{L}(X)\theta_i^j + \Lambda_k^j \theta_i^k. \quad (2.11)$$

The next step to construct the equivariant curvature is to define an operator $L_\Lambda : \Gamma(E) \rightarrow \Gamma(E)$ given by

$$L_\Lambda s = \Lambda s - \iota(X)Ds, \quad s \in \Gamma(E). \quad (2.12)$$

The matrix associated to this operator with respect to a local frame on U is

$$(L_\Lambda)_i^j = \Lambda_i^j - \theta_i^j(X). \quad (2.13)$$

Using (2.10) and the usual transformation rule for the connection matrix it is easy to check that $(L_\Lambda)_i^j$ is a tensorial matrix of the adjoint type: under a change of local frame one has

$$L'_\Lambda = g^{-1}L_\Lambda g.$$

We will compute now the covariant derivative of the matrix L_Λ . Using (2.11) we get:

$$\begin{aligned} DL_\Lambda &= dL_\Lambda + [\theta, L_\Lambda] \\ &= d\Lambda + [\theta, \Lambda] - (\mathcal{L}(X) - \iota(X)d)\theta - [\theta, \iota(X)\theta] \\ &= \iota(X)(d\theta + \theta \wedge \theta) = \iota(X)K, \end{aligned} \quad (2.14)$$

where K is the curvature matrix.

We can introduce now the *equivariant curvature* K_X for the vector bundle case, defined as follows:

$$K_X = K + uL_\Lambda. \quad (2.15)$$

This is not an equivariant differential form, not even a global differential form on M . To achieve this we have to introduce a symmetric invariant polynomial with r matrix entries, $P(A_1, \dots, A_r)$. Consider then the following quantity [13]:

$$P_X = P(K_X, \dots, K_X) = \sum_{i=0}^r u^i P_K^{(i)}, \quad (2.16)$$

where

$$P_K^{(i)} = \binom{r}{i} P(\overbrace{L_\Lambda, \dots, L_\Lambda}^i; K, \dots, K).$$

Notice that, as L_Λ is a tensorial matrix of the adjoint type, P_X is a globally defined differential form in $\Omega^*[u]$. Using (2.14) and the properties of symmetric invariant polynomials it is easy to prove that

$$\iota(X)P_K^{(i)} = dP_K^{(i+1)}, \quad (2.17)$$

and from this it follows that P_X is an equivariantly closed differential form on M .

2.3 EQUIVARIANT CURVATURE FOR PRINCIPAL BUNDLES

Let $\pi : P \rightarrow M$ be a principal bundle with group G . We suppose that we have two vector fields X_P, X acting on P and M , respectively. We will require that the one-parameter flow associated to $X_P, \hat{\phi}_t$, commutes with the right action of G on P :

$$\hat{\phi}_t(pg) = (\hat{\phi}_t p)g, \quad p \in P, \quad g \in G. \quad (2.18)$$

In this case, if ϕ_t is the one-parameter flow associated to X on M , we have $\pi \hat{\phi}_t =$

$\phi_t\pi$, and X and X_P are π -related. The vector field X_P is in addition right invariant:

$$(X_P)_{pg} = (R_g)_*p(X_P)_p. \quad (2.19)$$

Let θ be a connection one-form on P , and consider the function with values in the Lie algebra of G , \mathfrak{g} , given by $\theta(X_P) = \iota(X_P)\theta$. Using (2.19) and the properties of the connection it is immediate to see that $\theta(X_P)$ is a tensorial zero-form of the adjoint type, *i.e.*

$$\theta(X_P)_{pg} = \theta_{pg}((R_g)_*p(X_P)_p) = (\text{ad}g^{-1})\theta(X_P)_p. \quad (2.20)$$

Suppose now that the connection one-form verifies:

$$\mathcal{L}(X_P)\theta = 0. \quad (2.21)$$

This is the analog of having an equivariant connection for a vector bundle. When (2.21) holds we can construct an equivariant curvature for the principal bundle in a natural way. First, notice that the covariant derivative of $\theta(X_P)$ is given by:

$$D\theta(X_P) = -\iota(X_P)K, \quad (2.22)$$

where K is the curvature associated to θ . The equivariant curvature is defined as:

$$K_{X_P} = K - u\theta(X_P). \quad (2.23)$$

With the help of an invariant symmetric polynomial we can construct the form in $\Omega^*(P)[u]$:

$$P_{X_P} = P(K_{X_P}, \dots, K_{X_P}). \quad (2.24)$$

Taking into account (2.22) we can proceed as in the vector bundle case and show that P_{X_P} is an equivariantly closed differential form on P with respect to the action

of X_P . On the other hand, because of (2.20) and the usual arguments in Chern-Weil theory, P_{X_P} descends to a form in $\Omega^*(M)[u]$, \overline{P}_{X_P} . Recall that when a form ω on P descends to a form $\bar{\omega}$ on M , and the vector fields X_P on P and X on M are π -related, we have the following identities:

$$\begin{aligned}
\bar{\phi}(V_1, \dots, V_q) &= \phi(X_1, \dots, X_q), \\
(d\bar{\phi})(V_0, \dots, V_q) &= (d\phi)(X_0, \dots, X_q), \\
(\iota(X)\bar{\phi})(V_1, \dots, V_{q-1}) &= (\iota(X_P)\phi)(X_1, \dots, X_{q-1}), \\
(\mathcal{L}(X)\bar{\phi})(V_1, \dots, V_q) &= (\mathcal{L}(X_P)\phi)(X_1, \dots, X_q),
\end{aligned} \tag{2.25}$$

where the X_i are such that $\pi_* X_i = V_i$. It follows from (2.25) that, if P_{X_P} is equivariantly closed on P , then \overline{P}_{X_P} is equivariantly closed on M . We have therefore obtained an appropriate equivariant extension of the curvature of a principal bundle.

3. Equivariant extensions of the Thom form in the Mathai-Quillen formalism

We begin this section with a quick review of the Mathai-Quillen formalism. Most of the details will appear in the explicit constructions of the equivariant extensions of the Thom form for a vector field action, so we just recall some results. A more complete presentation can be found in [3, 4, 10].

3.1 THE MATHAI-QUILLEN FORMALISM

The Mathai-Quillen formalism [3] provides an explicit representative of the Thom form of a vector bundle E . Usually this form is introduced in the following way: consider an oriented vector bundle $\pi : E \rightarrow M$ with fibre $V = \mathbf{R}^{2m}$, equipped with an inner product g and a compatible connection D . Let P be the principal G -bundle over M such that E is the associated vector bundle. Then we can consider the G -equivariant cohomology of V in the Weil model, and we introduce the generators K and θ for the Weil complex $\mathcal{W}(\mathfrak{g})$, of degree two and one, respectively. As our vector bundle is oriented and has an inner product, we can reduce the structural group to $G = SO(2m)$. The universal Thom form U of Mathai and Quillen is an element in $\mathcal{W}(\mathfrak{g}) \otimes \Omega^*(V)$ given by:

$$U = (2\pi)^{-m} \text{Pf}(K) \exp\{-x_i x_i - (dx_i + \theta_{il} x_l)(K^{-1})_{ij}(dx_j + \theta_{jm} x_m)\}, \quad (3.1)$$

where x_i are orthonormal coordinate functions on V , and dx_i are their corresponding differentials. This expression includes the inverse of K , and in fact it should be properly understood, once the exponential is expanded, as:

$$\pi^{-m} e^{-x_i x_i} \sum_I \epsilon(I, I') \text{Pf}\left(\frac{1}{2} K_I\right) (dx + \theta x)^{I'}, \quad (3.2)$$

where I denotes a subset with an even number of indices, I' its complement and $\epsilon(I, I')$ the signature of the corresponding permutation. The equivalence of the two

representations is easily seen using Berezin integration. Of course, the expression (3.1) is easier to deal with, and in fact we can check its properties taking K^{-1} as a formal inverse of K . This is because we can consider (3.1) as an element of the ring of fractions with $\det(K)$ in the denominator. Being $\det(K)$ closed, we can extend the exterior derivative as an algebraic operator to this localization [3]. We will use later this principle to check the equivariantly closed character of our extension. One can also obtain a universal Thom form in the Cartan model of the G -equivariant cohomology, by putting the generator θ to zero. This gives an alternative representative which is useful in topological gauge theories [4, 10].

The form (3.1) can be mapped to a differential form in $\Omega(P \times V)$ using the Weil homomorphism. This amounts to substituting the algebraic generators of the Weil complex, K and θ , by the actual curvature and connection of the principal bundle P . The resulting form descends to E and gives an explicit representative of the Thom form of E , which will be denoted by $\Phi(E)$. If one uses the Cartan representative, one must enforce in addition a horizontal projection.

3.2 EQUIVARIANT EXTENSION OF THE THOM FORM: GENERAL CASE

One of the purposes of this paper is to find an equivariant extension of the Thom form for a vector field action, in the framework of the Mathai-Quillen formalism. If we look at the expressions for the equivariant extension of the curvature, (2.15) and (2.23), we see that they involve the contraction of the connection form with a vector field. It is clear that for the algebraic elements in the Weil algebra this operation is not defined, and therefore we won't work with the universal Thom form, but with the explicit Thom form as an element of $\Omega^{2m}(E)$. This has also the advantage of showing explicitly the geometry involved in the Mathai-Quillen formalism, which is sometimes hidden behind the use of G -equivariant cohomology.

Recall that we defined an “action” of a vector field on a vector bundle E as an operator acting on the space of sections of this bundle, and therefore not necessarily induced by an action of X_E on E . In this case we cannot consider the complex

$\Omega_{X_E}^*[u]$. However, given an invariant section s of this bundle, we can construct an equivariant extension of the pullback $s^*\Phi(E)$ on M .

In the framework of the Mathai-Quillen formalism we need an inner product on E , g , and a compatible connection D verifying:

$$d(g(s, t)) = g(Ds, t) + g(s, Dt), \quad s, t \in \Gamma(E) \quad (3.3)$$

Once we take into account the action of a vector field X on M , and the compatible operator on sections Λ , we need additional assumptions to construct our equivariant extensions. First of all, we assume, as in the previous subsection, that the connection D is equivariant. We also assume that the inner product is invariant with respect to the compatible actions:

$$\mathcal{L}(X)(g(s, t)) = g(\Lambda s, t) + g(s, \Lambda t), \quad s, t \in \Gamma(E) \quad (3.4)$$

From (3.3) and (3.4) one gets the following identity for the operator L_Λ defined in (2.12):

$$g(L_\Lambda s, t) + g(s, L_\Lambda t) = 0, \quad s, t \in \Gamma(E) \quad (3.5)$$

We suppose that our bundle E is orientable, and therefore we can reduce the structural group to $SO(2m)$ and consider orthonormal frames $\{s_i\}_{i=1, \dots, 2m}$ such that $g(s_i, s_j) = \delta_{ij}$. With respect to an orthonormal frame, the connection and curvature matrices are antisymmetric, and because of (3.5) L_Λ and K_X are antisymmetric too.

Consider now a trivializing open covering of M , $\{U_\alpha\}$, and the corresponding orthonormal frames $\{s_i^\alpha\}$. Let $s \in \Gamma(E)$ be an invariant section. Then, the following form is an equivariantly closed differential form on M and is an equivariant extension of the pullback of the Thom class by s :

$$s^*\Phi(E)_X^\alpha = (2\pi)^{-m} \text{Pf}(K_X) \exp\{-\xi_i^\alpha \xi_i^\alpha - (d\xi_i^\alpha + \theta_{il}^\alpha \xi_l^\alpha)(K_X^\alpha)_{ij}^{-1}(d\xi_j^\alpha + \theta_{jm}^\alpha \xi_m^\alpha)\}, \quad (3.6)$$

where $s = \xi_i s_i^\alpha$ is the local expression of s in U_α , and θ^α and K_X^α are respectively

the connection and the equivariant curvature matrices (the equivariant curvature is the one given in (2.15)). Both are defined with respect to the orthonormal frame $\{s_i^\alpha\}$.

To prove our statement, we will show first of all that the $s^*\Phi(E)_X^\alpha$ define a global differential form on M , *i.e.*, we will consider a change of trivialization on the intersections $U_\alpha \cap U_\beta$. The transformations of the different functions appearing here are:

$$\begin{aligned} s^\beta &= s^\alpha g_{\alpha\beta}, & \xi^\beta &= g_{\alpha\beta}^{-1} \xi^\alpha, \\ \theta^\beta &= g_{\alpha\beta}^{-1} \theta^\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}, \\ K_X^\beta &= g_{\alpha\beta}^{-1} K_X^\alpha g_{\alpha\beta}, \end{aligned} \tag{3.7}$$

where $g_{\alpha\beta}$ are the transition functions and take values in $SO(2m)$. To check the invariance of (3.6) under this transformation, notice that $\text{Pf}(K_X)$ is an invariant symmetric polynomial for antisymmetric matrices and therefore the results of section 2 hold. Also notice that $d\xi_i^\alpha + \theta_{il}^\alpha \xi_l^\alpha$ transforms as a tensorial matrix of the adjoint type (because it is the local expression of the covariant derivative Ds). It is easily checked that $s^*\Phi(E)_X^\alpha$ equals $s^*\Phi(E)_X^\beta$ on the intersections $U_\alpha \cap U_\beta$, and therefore the expression (3.6) defines a global differential form on M .

To prove that this differential form is in the kernel of d_X it is enough to do it for the local expression in (3.6), as d_X is a local operator. Again, by the results of section 2, $\text{Pf}(K_X)$ is already equivariantly closed, and we only need to check this property for the exponent in (3.6). The computation is lengthy but straightforward. Recall that s is an invariant section, and locally this can be written as:

$$\Lambda(\xi_i s_i) = X(\xi_i) s_i + \xi_i \Lambda_{ji} s_j = 0. \tag{3.8}$$

It follows then that

$$d_X d\xi_i = -uX(\xi_i) = -u\Lambda_{ij} \xi_j, \tag{3.9}$$

and we get the following expression:

$$\begin{aligned}
& d_X \{ \xi_i \xi_i + (d\xi_i + \theta_{il} \xi_l)(K_X^{-1})_{ij}(d\xi_j + \theta_{jm} \xi_m) \} \\
&= 2d\xi_i \xi_i + [u(\Lambda_{il} \xi_l - \theta_{il}(X) \xi_l) + d\theta_{il} \xi_l - \theta_{il} \xi_l](K_X^{-1})_{ij}(d\xi_j + \theta_{jm} \xi_m) \\
&- (d\xi_i + \theta_{il} \xi_l)(d_X K_X^{-1})_{ij}(d\xi_j + \theta_{jm} \xi_m) \\
&- (d\xi_i + \theta_{il} \xi_l)(K_X^{-1})_{ij}[u(\Lambda_{jm} \xi_m - \theta_{jm}(X) \xi_m) + d\theta_{jm} \xi_m - \theta_{jm} \xi_m].
\end{aligned} \tag{3.10}$$

If we add to this $(\theta_{il} + \theta_{li})(d\xi_l + \theta_{lp} \xi_p)(K_X^{-1})_{ij}(d\xi_j + \theta_{jm} \xi_m) = 0$, and we take into account that

$$D(K_X^{-1})_{ij} = d(K_X^{-1})_{ij} + \theta_{il}(K_X^{-1})_{lj} - (K_X^{-1})_{il}\theta_{lj}, \tag{3.11}$$

then (3.10) reads:

$$\begin{aligned}
& 2d\xi_i \xi_i + (K_X)_{il} \xi_l (K_X^{-1})_{ij}(d\xi_j + \theta_{jm} \xi_m) \\
& - (d\xi_i + \theta_{ip} \xi_p)[D(K_X^{-1})_{ij} - u\iota(X)(K_X^{-1})_{ij}](d\xi_j + \theta_{jm} \xi_m) \\
& + (d\xi_i + \theta_{ip} \xi_p)(K_X^{-1})_{il}(K_X)_{lm} \xi_m.
\end{aligned} \tag{3.12}$$

We can compute $DK_X^{-1} - u\iota(X)K_X^{-1}$ considering K_X^{-1} a formal inverse of K_X . Notice first that, because of the Bianchi identity and (2.14), we have:

$$DK_X = u\iota(X)K, \quad d_X K_X = -[\theta, K_X]. \tag{3.13}$$

As d_X extends to the ring of fractions with $\det K_X$ in the denominator (because $\det K_X$ is d_X -closed), we have:

$$d_X K_X^{-1} = K_X^{-1}[\theta, K_X]K_X^{-1} = -[\theta, K_X^{-1}], \tag{3.14}$$

and finally we get:

$$DK_X^{-1} - u\iota(X)K_X^{-1} = d_X K_X^{-1} + [\theta, K_X^{-1}] = 0 \tag{3.15}$$

Using (3.15) and the antisymmetry of the matrices $(K_X)_{ij}$, θ_{ij} , we see that (3.12) equals zero. Therefore, (3.6) is in the kernel of d_X , and according to (2.3) it is an

equivariantly closed differential form. It is clear that it is an equivariant extension of the pullback $s^*\Phi(E)$, because if we put $u = 0$ we recover the pullback of the Mathai-Quillen form.

3.3 EQUIVARIANT EXTENSION OF THE THOM FORM: VECTOR BUNDLE CASE

Now we will consider the case in which we have a vector field X_E acting on the vector bundle E , and the action Λ is the one induced from it. In this case it makes sense to construct an equivariant extension of the Thom form with respect to the X_E action. Again we will proceed locally and we will construct the extension on trivializing open sets $U_\alpha \times V$.

Let $\pi : E \rightarrow M$ be an orientable real vector bundle of rank $2m$ with an action of a vector field X_E compatible with an action of X on M in the sense of subsection 2.2. On the fibre $V = \mathbf{R}^{2m}$ we choose an orthonormal basis $\{e_i\}$ with respect to the standard inner product $(,)$ on it, and we denote by x_i the coordinate functions with respect to this basis. Let $\{U_\alpha\}$ be a trivializing open covering of M , with attached diffeomorphisms

$$\phi_\alpha : U_\alpha \times V \rightarrow \pi^{-1}(U_\alpha). \quad (3.16)$$

If g is the metric on E , we can reduce the structural group in such a way that $g(\phi_\alpha(m, v), \phi_\alpha(m, w)) = (v, w)$. This also gives an orthonormal frame for each U_α in the standard way:

$$s_i^\alpha(m) = \phi_\alpha(m, e_i). \quad (3.17)$$

We want to define a vector field action \hat{X}_α on each $U_\alpha \times V$ such that

$$(\phi_\alpha^{-1})_*(X_E) = \hat{X}_\alpha. \quad (3.18)$$

To do this we will define a one-parameter flow $\hat{\phi}_t$ inducing \hat{X}_α . The natural way is to use the conditions of compatibility of the vector field actions. On the first

factor, U_α , we use the restriction of one-parameter flow associated to X , and we take the appropriate t -interval for this map to be well defined. On the second factor we use the homomorphism between fibres given by the one-parameter flow associated to X_E , ϕ_t^E . Written in a local trivialization, this homomorphism means that, if $p \in E_m$, $\phi_t^E p \in E_{\phi_t m}$, then

$$(\pi_2 \phi_\alpha^{-1})(\phi_t^E p) = \lambda(t, m) \pi_2 \phi_\alpha^{-1}(p), \quad (3.19)$$

where π_2 denotes the projection of ϕ_α^{-1} on the second factor and λ is an endomorphism of V which depends on t , the basepoint m and the trivialization. Now we can define:

$$\hat{\phi}_t(m, v) = (\phi_t(m), \lambda(t, m)v), \quad (m, v) \in U_\alpha \times V. \quad (3.20)$$

Notice that the endomorphism λ verifies:

$$\lambda(s, \phi_t(m)) \lambda(t, m) = \lambda(s + t, m). \quad (3.21)$$

From the definition of $\hat{\phi}_t$ it follows that

$$\phi_\alpha^{-1} \phi_t^E = \hat{\phi}_t \phi_\alpha^{-1}, \quad (3.22)$$

and this in turn implies (3.18).

The procedure is now similar to the one presented in the preceding section. We define the following form on $\Omega^*(U_\alpha \times V)[u]$:

$$\Phi(E)_X^\alpha = (2\pi)^{-m} \text{Pf}(K_X) \exp\{-x_i x_i - (dx_i + \theta_{il} x_l)(K_X^\alpha)_{ij}^{-1}(dx_j + \theta_{jm} x_m)\}, \quad (3.23)$$

where θ_{ij} , $(K_X)_{ij}$ denote respectively the connection and equivariant curvature matrices associated to the orthonormal frame defined in (3.17). The index α labeling the trivialization is understood. We want to check that (3.23) defines a global differential form on E . First we will consider the behavior of $\omega^\alpha = \Phi(E)_X^\alpha$ under a

change of trivialization. The transition functions for the vector bundle are defined as $g_{\beta\alpha} = \phi_\beta^{-1}\phi_\alpha$, restricted as usual to $\{x\} \times V$. The behavior of the connection and curvature matrices under the change of trivialization is given in (3.7), and the gluing conditions for the elements in the trivializing open sets are

$$(m, v)^\beta = (m, g_{\alpha\beta}^{-1}(v))^\alpha. \quad (3.24)$$

The coordinate functions then transform as $x_i \rightarrow (g_{\alpha\beta}^{-1})_{ij}x_j$. Following the same steps as in the preceding section we see that the forms ω^α do not change when we go from the α description to the β description:

$$g_{\alpha\beta}^*\omega^\alpha = \omega^\beta. \quad (3.25)$$

The forms ω^α define the corresponding forms on $\pi^{-1}(U_\alpha)$ by taking $(\phi_\alpha^{-1})^*\omega^\alpha$ on these open sets. On the intersections we have, because of (3.25),

$$(\phi_\alpha^{-1})^*\omega^\alpha = (\phi_\beta^{-1})^*\omega^\beta, \quad (3.26)$$

and therefore they define a global differential form on E . Now it is clear that, if the ω^α are in the kernel of $d_{\hat{X}}$, the $(\phi_\alpha^{-1})^*\omega^\alpha$ are in the kernel of d_{X_E} . This is a consequence of the following simple result: if $f : M \rightarrow N$ is a differentiable map, $\omega \in \Omega^*(N)$, and X_M, X_N are two vector fields which are f -related, then

$$\iota(X_M)f^*\omega = f^*\iota(X_N)\omega. \quad (3.27)$$

Using (3.27) and (3.18) we see that

$$d_{X_E}(\phi_\alpha^{-1})^*\omega^\alpha = (\phi_\alpha^{-1})^*(d - \iota((\phi_\alpha^{-1})_*(X_E)))\omega^\alpha = (\phi_\alpha^{-1})^*(d_{\hat{X}}\omega^\alpha). \quad (3.28)$$

To prove that the ω^α are in the kernel of $d_{\hat{X}}$, notice that the computation is very similar to the one presented in the preceding section. The only new thing we must

compute is $d_{\hat{X}}(dx_i) = -u\mathcal{L}(\hat{X})x_i$. Using the definition of Lie derivative and the action of the one-parameter group associated to \hat{X} and given in (3.20), we get:

$$(\mathcal{L}(\hat{X})x_i)(m, v) = -\frac{d}{dt}\lambda_{ij}(-t, m)\Big|_{t=0}x_j(v). \quad (3.29)$$

The matrix appearing in this expression is not new. To see it, notice that the matrix representation of the operator Λ with respect to the orthonormal frame (3.17) is given by

$$(\Lambda s_i^\alpha)(m) = \lim_{t \rightarrow 0} \frac{s_i^\alpha(m) - \phi_t^E s_i^\alpha(\phi_{-t}m)}{t}. \quad (3.30)$$

Using (3.17) and (3.22) we obtain:

$$\phi_t^E s_i^\alpha(\phi_{-t}m) = \phi_\alpha \hat{\phi}_t(\phi_{-t}m, e_i) = s_j^\alpha(m)\lambda_{ji}(t, \phi_{-t}m), \quad (3.31)$$

and this gives

$$(\Lambda s_i^\alpha)(m) = -s_j^\alpha(m)\frac{d}{dt}\lambda_{ji}(t, \phi_{-t}m)\Big|_{t=0} \quad (3.32)$$

Finally, using (3.21) and comparing (3.29) and (3.32) we get

$$(\mathcal{L}(\hat{X})x_i)(m, v) = -\Lambda_{ij}(m)x_j(v). \quad (3.33)$$

If we compare this expression to (3.9) we see that the computation of the equivariant exterior derivative of ω^α with respect to \hat{X} simply mimicks the one we did in the preceding section. Therefore, the forms defined in (3.23) are in the kernel of $d_{\hat{X}}$ and the global differential form $\Phi(E)_X$ they induce on E is an equivariantly closed differential form because of (3.28). It clearly equivariantly extends the Mathai-Quillen expression for the Thom form of the bundle.

Consider now an invariant section $s \in \Gamma(E)$. Because of (2.8) and (3.27) it is easy to see that $s^*\Phi(E)_X$ is an equivariantly closed differential form on the base

manifold M . Of course the local expression of this form coincides with (3.6): the map $\phi^\alpha s : U_\alpha \rightarrow U_\alpha \times V$ is given by

$$(\phi^\alpha s)(m) = (m, \xi_i^\alpha(m)e_i), \quad (3.34)$$

where we wrote $s = \xi_i^\alpha s_i^\alpha$. As the local expression of $s^* \Phi(E)_X$ is

$$(s^* \Phi(E)_X)^\alpha = (\phi^\alpha s)^* \Phi(E)_X^\alpha, \quad (3.35)$$

and from (3.34) this amounts to substitute x_i by ξ_i^α in (3.23), we recover precisely (3.6).

Finally, we will give a field theory expression for $\Phi(E)_X$ using Berezin integration. Introduce Grassmann variables ρ_i for the local coordinates of the fibre. The standard rules of Berezin integration [3, 10] give the following representative for the local expression (3.23):

$$\Phi(E)_X^\alpha = \pi^{-m} e^{-x_i x_i} \int \mathcal{D}\rho \exp\left(\frac{1}{4}\rho_i K_{ij} \rho_j + \frac{u}{4}\rho_i (L_\Lambda)_{ij} \rho_j + i(dx_i + \theta_{ij} x_j)\rho_i\right). \quad (3.36)$$

With this expression at hand, one can also introduce the standard objects in topological field theory, namely a gauge fermion and a BRST complex. Following [10], we introduce an auxiliary field π_i with the meaning of a basis of differential forms dx_i for the fibre. The BRST operator is given by the $d_{\hat{X}}$ cohomology, and therefore we have:

$$Q\rho_i = \pi_i, \quad Q\pi_i = u\Lambda_{ij}\pi_j. \quad (3.37)$$

On the original fields x_i and the matrix-valued functions on U_α , θ_{ij} , K_{ij} , $(L_\Lambda)_{ij}$, Q acts again as $d_{\hat{X}}$. The gauge fermion is the same than the gauge fermion in the Weil model for the Mathai-Quillen formalism [10]:

$$\Psi = -\rho_i(ix_i - \frac{1}{4}\theta_{ij}\rho_j + \frac{1}{4}\pi_i), \quad (3.38)$$

and it is easily checked that $Q\Psi$ gives, after integrating out the auxiliary field π_i , the exponent in (3.36). This representative will be useful to construct the equivariant

extension for topological sigma models. Notice that in the expression (3.36) we can work with a non-orthonormal metric on V by introducing the corresponding jacobian in the integration measure.

3.4 EQUIVARIANT EXTENSION OF THE THOM FORM: PRINCIPAL BUNDLE CASE

We will consider, finally, the case in which the vector bundle E is explicitly given as an associated vector bundle to a principal bundle $\pi : P \rightarrow M$, *i.e.*, we consider the action of the structural group G on $P \times V$ given by $(p, v)g = (pg, g^{-1}v)$, and we form the quotient $E = P \times V/G$. Notice that $P \times V$ can be considered as a principal bundle over E . We assume that we have a vector field action on $P \times V$ whose one-parameter flow μ_t has the following structure:

$$\mu_t(p, v) = (\phi_t^P p, \lambda(t, p)v) \quad p \in P, v \in V, \quad (3.39)$$

where $\lambda(t, p)$ is an endomorphism of V . We also assume that this flow conmmutes with the G -action on $P \times V$:

$$(\phi_t^P p)g = \phi_t^P(pg), \quad \lambda(t, pg) = g^{-1}\lambda(t, p)g. \quad (3.40)$$

Because of the above condition, a vector field action on E is induced in the natural way, and the one-parameter flow ϕ_t^P gives in turn a vector field action on $M = P/G$ in the way considered in subsection 2.3, with one-parameter flow ϕ_t . In addition, with these assumptions, the vector field action on E is compatible with the vector field action on M according to our definition in subsection 2.2. Condition (i) is immediate, and to see that condition (ii) holds consider a trivializing open covering for M , $\{U_\alpha\}$, and the corresponding map $\nu_\alpha : \pi^{-1}(U_\alpha) \rightarrow G$. If $m \in U_\alpha$, $\phi_t(m) \in U_\beta$, the map between the fibres $E_m, E_{\phi_t(m)}$ is given by the homomorphism

$$\nu_\beta(\phi_t^P p)\lambda(t, p)\nu_\alpha(p)^{-1}, \quad (3.41)$$

where $p \in \pi^{-1}(m)$. Using (3.40) it is easy to see that (3.41) only depends on the basepoint m and t . The vector fields on P, E and M will be denoted, respectively,

by X_P , X_E and X . Our last assumption is that there is an inner product $(,)$ on V preserved by both the action of G and the endomorphisms $\lambda(t, p)$. As usual, this means that the matrix

$$\Lambda_{ij}(p) = \lim_{t \rightarrow 0} \frac{1}{t} [\delta_{ij} - \lambda(t, p)_{ij}] \quad (3.42)$$

is antisymmetric, where the components are taken with respect to an orthonormal basis e_i of V . If we regard $P \times V$ as a principal bundle, the second condition in (3.40) imply that Λ is a tensorial matrix of the adjoint type.

We will be particularly interested in the case in which $\lambda(t, p)$ doesn't depend on p . In this case we have that Λ is a constant matrix commuting with all the $g \in G$ (and then with all the elements in the Lie algebra \mathfrak{g}). This happens, for instance, if $G = U(m) \subset SO(2m)$ and Λ has the structure:

$$\Lambda = \begin{pmatrix} 0 & 1 & \dots & & \\ -1 & 0 & \dots & & \\ \vdots & \vdots & \ddots & & \\ & & \dots & 0 & 1 \\ & & \dots & -1 & 0 \end{pmatrix}. \quad (3.43)$$

This is in fact the situation we will find in the application of our formalism to non-abelian monopoles on four-manifolds.

Let θ and K be respectively the connection and curvature of P . Assume now, as in subsection 2.3, that $\mathcal{L}(X_P)\theta = 0$, and that Λ is a constant matrix commuting with all the $A \in \mathfrak{g}$. Then $D\Lambda = 0$. We want to construct an equivariantly closed differential form on $P \times V$ with respect to the vector field action $\hat{X} = (X_P, X_V)$, where X_V is associated to the flow $\lambda(t)$. First of all we define an equivariant curvature on $P \times V$:

$$K_X = K + u(\Lambda - \theta(X_P)). \quad (3.44)$$

Notice that $\Lambda - \theta(X_P)$ is a tensorial matrix of the adjoint type, and if $P(A_1, \dots, A_r)$ is an invariant symmetric polynomial for the adjoint action of \mathfrak{g} , then we can go

through the arguments of subsection 2.3 to show that $P(K_X, \dots, K_X)$ defines an equivariantly closed differential form on $P \times V$. The construction of the equivariant extension of the Thom class is very similar to the ones we have done before, but now we define a form on $P \times V$ and we will show that it descends to E . Consider then the following element in $\Omega^*(P \times V)[u]$:

$$\Phi(P \times V) = (2\pi)^{-m} \text{Pf}(K_X) \exp\{-x_i x_i - (dx_i + \theta_{il} x_l)(K_X^\alpha)_{ij}^{-1}(dx_j + \theta_{jm} x_m)\}, \quad (3.45)$$

where x_i are, as before, orthonormal coordinates on the fibre V . First we will check that the above form descends to E . For this we must check that it is right invariant and that it vanishes on vertical fields. The first property is easily checked using the expressions:

$$(R_g^* x_i)(v) = x_i(g^{-1}v) = g_{ij}^{-1} x_j(v), \quad R_g^* dx_i = g_{ij}^{-1} dx_j. \quad (3.46)$$

To check the horizontal character, notice that K_X is horizontal (for K is and $\Lambda - \theta(X_P)$ is a zero-form), and then we only have to check it for $dx_i + \theta_{il} x_l$, as in [3]. Notice that we are considering $P \times V$ as a principal bundle over E , and therefore a fundamental vector field A^* (corresponding to $A \in \mathfrak{g}$) is induced by the G -action on both factors. Using the properties of the connection one-form and the action of G on V , one immediately gets:

$$\iota(A^*)\theta_{ij} = A_{ij}, \quad \iota(A^*)dx_i = \mathcal{L}(A^*)x_i = -A_{ij}x_j. \quad (3.47)$$

We see then that $\Phi(P \times V)$ descends to E . This also simplifies the computation of $d_{\hat{X}}\Phi(P \times V)$. First, we define a connection on $P \times V$ by pulling-back the connection on P . The horizontal subspace at (p, v) is given by $H_p \oplus V$, where H_p is the horizontal subspace of $T_p P$. If we denote by Φh the horizontal projection of a form Φ on $P \times V$ that descends to E , we have:

$$d\Phi = d\Phi h = D\Phi, \quad \iota(\hat{X})\Phi = (\iota(\hat{X})\Phi)h. \quad (3.48)$$

As θ vanishes on horizontal vectors, we can put it to zero after computing the exterior derivative of (3.45). Also notice that the covariant derivative defined by

the pullback connection on $P \times V$ acts as the covariant derivative of P on the differential forms in $\Omega^*(P)$, and as the usual exterior derivative on the forms in $\Omega^*(V)$.

Now we can compute $d_{\hat{X}}\Phi(P \times V)$ in a simple way. Again we only need to compute the equivariant exterior derivative of the exponent:

$$\begin{aligned}
& d_{\hat{X}} \{-x_i x_i - (dx_i + \theta_{il} x_l)(K_X)_{ij}^{-1}(dx_j + \theta_{jm} x_m)\} \\
&= 2dx_i x_i + [K_{il} x_l - u(\mathcal{L}(X_V)x_i + \theta_{il} x_l)](K_X)_{ij}^{-1} dx_j \\
&\quad - dx_i [(DK_X^{-1})_{ij} - u\iota(X_P)(K_X)_{ij}^{-1}] dx_j \\
&\quad - dx_i (K_X)_{ij}^{-1} [K_{jp} x_p - u(\mathcal{L}(X_V)x_j + \theta_{jp} x_p)].
\end{aligned} \tag{3.49}$$

The computation of $\mathcal{L}(X_V)x_i$ is straightforward from the definition (3.39) and one obtains $-\Lambda_{ij}x_j$ as in (3.33). Assuming (2.21) we get $DK_X = u\iota(X_P)K$ and therefore, using the same arguments leading to (3.15), we see that (3.49) is zero. If we denote by $\tilde{\pi}$ the projection of $P \times V$ on E , it follows from our assumptions that $\tilde{\pi}_*\hat{X} = X_E$, and therefore, using (2.25) we see that the form induced by (3.45) on E is equivariantly closed with respect to X_E .

The above computation also shows the possibility of introducing a Cartan-like formulation of the equivariant extension we have obtained. Consider the form on $\Omega^*(P \times V)[u]$ given by

$$\Phi(P \times V)_C = (2\pi)^{-m} \text{Pf}(K_X) \exp\{-x_i x_i - dx_i (K_X^\alpha)_{ij}^{-1} dx_j\}. \tag{3.50}$$

Clearly it is still invariant under the action of G , but the horizontal character fails. However we can consider the horizontal projection of this form, $\Phi(P \times V)_C h$, where the horizontal subspace is defined as before by the pullback connection. This form coincides in fact with (3.45), because the horizontal projection only applies to dx_i and gives

$$(dx_i)h = dx_i + \theta_{ij}x_j. \tag{3.51}$$

The interesting thing about (3.50) is that when one enforces the horizontal projection as in [4], one obtains the adequate formalism to topological gauge theories.

We will then follow this procedure to obtain a representative which will be useful later.

We suppose now that we have a metric g on P which is G -invariant. We use this metric to define the connection on P , by declaring the horizontal subspace to be the orthogonal complement of the vertical one. More explicitly, one starts from the map defining fundamental vector fields on P :

$$C_p = R_{p*} : \mathfrak{g} \rightarrow T_p P. \quad (3.52)$$

Consider now the following differential form on P with values in \mathfrak{g}^* :

$$\nu_p(Y_p, A) = g_p(R_{p*} A, Y_p), \quad Y_p \in T_p P, \quad A \in \mathfrak{g}.$$

If we denote by C_p^\dagger the adjoint of C_p (which is defined by the metric on P together with the Killing form on \mathfrak{g}), and let $R = C^\dagger C$, the connection one-form is defined by:

$$\theta = R^{-1} \nu. \quad (3.53)$$

With the assumptions we have made concerning P , the condition $\mathcal{L}(X_P)\theta = 0$ is equivalent to the metric being invariant under the vector field action. Now we will write (3.50) as a fermionic integral over Grassmann variables:

$$\Phi(P \times V)_C = (\pi)^{-m} e^{-x_i x_i} \int \mathcal{D}\rho \exp\left(\frac{1}{4} \rho_i (K_X)_{ij} \rho_j + i dx_i \rho_i\right). \quad (3.54)$$

As we want to make a horizontal projection of this form, we can write $K = d\theta = R^{-1} d\nu$, and for the equivariant curvature defined in (3.44) we have:

$$K_X = R^{-1}(d\nu - u\nu(X_P)) + u\Lambda. \quad (3.55)$$

If we introduce Lie algebra variables λ, ϕ and use the Fourier inversion formula of

[4], we get the expression:

$$\begin{aligned} \Phi(P \times V)_C = & (2\pi)^{-d}(\pi)^{-m} e^{-x_i x_i} \int \exp \left(\frac{1}{4} \rho_i (\phi_{ij} + u \Lambda_{ij}) \rho_j + i dx_i \rho_i \right. \\ & \left. + i \langle d\nu - u\nu(X_P), \lambda \rangle - i \langle \phi, R\lambda \rangle \right) \det R \mathcal{D}\rho \mathcal{D}\phi \mathcal{D}\lambda, \end{aligned} \quad (3.56)$$

where \langle, \rangle denotes the Killing form of \mathfrak{g} , and $d = \dim G$. Notice that in this expression the integration over λ gives a δ -function constraining ϕ to be $K - \theta(X_P)$, which is precisely (2.23), the equivariant curvature of the principal bundle P . To enforce the horizontal projection, we multiply by the normalized invariant volume form $\mathcal{D}g$ along the G -orbits, and we can write [4, 10]:

$$(\det R) \mathcal{D}g = \int \mathcal{D}\eta \exp i \langle \eta, \nu \rangle, \quad (3.57)$$

where η is a fermionic Lie algebra variable. Putting everything together we obtain a representative for the horizontal projection:

$$\begin{aligned} \Phi(P \times V)_C h = & (2\pi)^{-d}(\pi)^{-m} e^{-x_i x_i} \int \exp \left(\frac{1}{4} \rho_i (\phi_{ij} + u \Lambda_{ij}) \rho_j + i dx_i \rho_i \right. \\ & \left. + i \langle d\nu - u\nu(X_P), \lambda \rangle - i \langle \phi, R\lambda \rangle + i \langle \eta, \nu \rangle \right) \mathcal{D}\eta \mathcal{D}\rho \mathcal{D}\phi \mathcal{D}\lambda, \end{aligned} \quad (3.58)$$

where integration over the fibre is understood.

We will introduce now a BRST complex in a geometrical way. As in the preceding section, we introduce auxiliary fields π_i with the meaning of a basis of differential forms for the fibre. The natural BRST operator is precisely the $d_{\hat{X}}$ operator, but we must take into account that we have in $\Phi(P \times V)_C h$ is the horizontal projection of dx_i , given in (3.51). Acting with the equivariant exterior derivative and projecting horizontally, as we did in (3.49), we get:

$$d_{\hat{X}}(dx_i h) = u \Lambda_{ij} x_j + (K_{ij} - u \theta_{ij}(X_P)) x_j. \quad (3.59)$$

Remembering that ϕ is equivalent to the equivariant curvature of P , the BRST

operator for the fibre is naturally given by:

$$Q\rho_i = \pi_i, \quad Q\pi_i = (u\Lambda_{ij} + \phi_{ij})\rho_j. \quad (3.60)$$

Following [10] we introduce a “localizing” and a “projecting” gauge fermion:

$$\Psi_{\text{loc}} = -\rho_i(ix_i + \frac{1}{4}\pi_i), \quad \Psi_{\text{proj}} = i\langle\lambda, \nu\rangle. \quad (3.61)$$

On the Lie algebra elements the BRST operator acts as:

$$Q\lambda = \eta, \quad Q\eta = -[\phi, \lambda]. \quad (3.62)$$

In order to obtain (3.58) from (3.61) using the BRST complex, we must also take into account the horizontal projection of forms on P , like in (3.59), and the equivariant exterior derivative is then given as

$$d - \iota(C\phi) - u\iota(X_P). \quad (3.63)$$

Notice that ϕ is an element of the Lie algebra \mathfrak{g} , and therefore $C\phi$ is a fundamental vector field on P . Using (3.62) and (3.63) as BRST operators acting on the gauge fermions (3.61), the topological lagrangian (3.58) corresponding to an equivariant extension of the Thom form is recovered. The BRST complex we have introduced looks like a $G \times X_P$ equivariant cohomology, but one shouldn’t take this analogy too seriously. If one formulates this equivariant cohomology in the Weil model, the relation $\iota(X_P)\theta = 0$ should be introduced. Clearly, this is not true geometrically unless X_P is horizontal. In fact, this term appears in the equivariant curvature of the principal bundle, and therefore in the expression for ϕ once the δ -function constraint has been taken into account.

The last point we would like to consider is the pullback of the equivariant extension we have obtained for this case. As (3.45) descends to a equivariantly

closed differential form on E , we can pull it back through an invariant section $\hat{s} : M \rightarrow E$ as we did in subsection 3.3. But recall that every section of E is associated to a G -equivariant map

$$s : P \rightarrow V, \quad s(pg) = g^{-1}s(p). \quad (3.64)$$

If \hat{s} is invariant, then the corresponding s is (3.64) verifies:

$$s\phi_t^P = \lambda(t)s. \quad (3.65)$$

Consider now the map $\tilde{s} : P \rightarrow P \times V$ given by $\tilde{s}(p) = (p, s(p))$. From the above it follows that $\tilde{s}^*\Phi(P \times V)$ is a closed equivariant differential form on P with respect to X_P , and in fact it descends to M , producing the same form we would get had we used the section \hat{s} . We have then the commutative diagram:

$$\begin{array}{ccc} \Omega^*(P \times V)_{\text{basic}, E} & \longrightarrow & \Omega^*(E) \\ \tilde{s}^* \downarrow & & \downarrow \hat{s}^* \\ \Omega^*(P)_{\text{basic}, M} & \longrightarrow & \Omega^*(M) \end{array} \quad (3.66)$$

This diagram should be kept in mind in topological gauge theories, where the topological lagrangian is usually a basic form on P descending to M . When considering the equivariant extension of the Mathai-Quillen form we will have the same situation, with an equivariantly closed differential form on P descending to M .

4. Applications

4.1 TOPOLOGICAL SIGMA MODELS

Applying the previous formalism to the topological sigma model [2] we will obtain the model of [16], which was constructed by twisting an $N = 2$ supersymmetric sigma model with potentials [20]. The Mathai-Quillen formalism for usual sigma models can be found in [8, 9, 10].

Let M be an almost hermitean manifold on which a vector field X acts preserving the almost complex structure J and the hermitean metric G :

$$\mathcal{L}(X)J = \mathcal{L}(X)G = 0. \quad (4.1)$$

We have then a one-parameter flow ϕ_t associated to X which is almost complex with respect to J :

$$\phi_{t*}J = J\phi_{t*}. \quad (4.2)$$

Let Σ be a Riemann surface with a complex structure ϵ and metric h inducing ϵ . In the topological sigma model, formulated in the framework of the Mathai-Quillen formalism, one takes as the base manifold \mathcal{M} the space of maps

$$\mathcal{M} = \text{Map}(\Sigma, M) = \{f : \Sigma \rightarrow M, f \in C^\infty(\Sigma, M)\}. \quad (4.3)$$

Given a $f \in \mathcal{M}$ we can consider the bundle over Σ given by $T^*\Sigma \otimes f^*TM$, and define a bundle over \mathcal{M} by giving the fibre on $f \in \mathcal{M}$:

$$\mathcal{V}_f = \Gamma(T^*\Sigma \otimes f^*TM)^+, \quad (4.4)$$

where $+$ denotes the self-duality constraint for the elements $\rho \in \mathcal{V}_f$:

$$J\rho\epsilon = \rho. \quad (4.5)$$

There is a natural way to define a vector field action on \mathcal{M} induced by the action

of X on M :

$$(\phi_t f)(\sigma) = \phi_t(f(\sigma)), \quad (4.6)$$

and similarly we can define an action on the fibre \mathcal{V}_f :

$$(\tilde{\phi}_t \rho)(\sigma) = \phi_{t*}(\rho(\sigma)). \quad (4.7)$$

This action is well defined, *i.e.*, $\tilde{\phi}_t \rho$ verifies the self-duality constraint (4.5) when ρ does, due to (4.2). It is also clear that the compatibility conditions of subsection 2.2 hold: first, $(\tilde{\phi}_t \rho)(\sigma)$ takes values in $T_\sigma^* \Sigma \otimes T_{\phi_t f(\sigma)} M$, therefore $\tilde{\phi}_t \rho \in \mathcal{V}_{\phi_t f}$; second, the map (4.7) is clearly a linear map between fibres, as it is given by the action of ϕ_{t*} .

Now we will define metrics on \mathcal{M} and \mathcal{V} . Let Y, Z vector fields on \mathcal{M} . We can formally define a local basis on $T\mathcal{M}$ from a local basis on M , given by functional derivatives with respect to the coordinates: $\delta/\delta f^\mu(\sigma)$ [9]. A vector field on \mathcal{M} will be written locally as:

$$Y = \int d^2 \sigma Y^\mu(f(\sigma)) \frac{\delta}{\delta f^\mu(\sigma)}. \quad (4.8)$$

With respect to this local coordinate description we define the metric on \mathcal{M} as:

$$(Y, Z) = \int d^2 \sigma \sqrt{h} G_{\mu\nu} Y^\mu(f(\sigma)) Z^\nu(f(\sigma)). \quad (4.9)$$

In a similar way, if $\rho, \tau \in \mathcal{V}_f$ have local coordinates $\rho_\alpha^\mu, \tau_\beta^\nu$, the metric on \mathcal{V}_f is given by:

$$(\rho, \tau) = \int d^2 \sigma \sqrt{h} G_{\mu\nu} h^{\alpha\beta} \rho_\alpha^\mu \tau_\beta^\nu. \quad (4.10)$$

As X is a Killing vector for the hermitean metric G , both (4.9) and (4.10) verify (3.4). Now we will define a connection on \mathcal{V} compatible with (4.10). In analogy

with the local basis for $T\mathcal{M}$, we can construct a local basis of differential forms on $\Omega^*(\mathcal{M})$, $\tilde{d}f^\mu(\sigma)$, which is dual to $\delta/\delta f^\mu(\sigma)$ in a functional sense:

$$(\tilde{d}f^\mu(\sigma))\left(\frac{\delta}{\delta f^\nu(\sigma')}\right) = \delta_\nu^\mu \delta(\sigma - \sigma'). \quad (4.11)$$

Let s be a section of \mathcal{V} , with local coordinates s_α^μ . We will define the connection by the local expression:

$$Ds_\alpha^\mu = \tilde{d}s_\alpha^\mu + \left(\Gamma_{\nu\lambda}^\mu + \frac{1}{2}D_\nu J_\kappa^\mu J_\lambda^\kappa\right)s_\alpha^\lambda \tilde{d}f^\nu, \quad (4.12)$$

where \tilde{d} is the exterior derivative on \mathcal{M} , with local expression:

$$\tilde{d}s_\alpha^\mu = \int d^2\sqrt{h} \frac{\delta s_\alpha^\mu}{\delta f^\nu(\sigma)} \tilde{d}f^\nu(\sigma). \quad (4.13)$$

The connection defined in this way is induced by the connection on M given by:

$$D = D_G + \frac{1}{2}D_G J J, \quad (4.14)$$

where D_G is the Riemannian connection canonically associated to the hermitean metric G on M . Notice that, if M is Kähler, then $D_G J = 0$ and the covariant derivative reduces to the usual form. It is easy to see that (4.12) is compatible both with the self-duality constraint and with the metric (4.10).

To define the usual topological sigma model we also need a section of \mathcal{V} . This section is essentially the Gromov equation for pseudoholomorphic maps $\Sigma \rightarrow M$, and can be written as:

$$s(f) = f_* + J f_* \epsilon. \quad (4.15)$$

Using (4.5) it is easy to show that s is invariant under the vector field action on \mathcal{M} . The last ingredient we need to construct the equivariant extension of the Thom form is to check the equivariance of the connection (4.12). As the action of the

vector field X on \mathcal{M} is induced by the corresponding action on M , it is sufficient to prove the equivariance of the connection (4.14) (equivalently, if we check the equivariance in local coordinates for \mathcal{M} , \mathcal{V} , we are reduced to a computation involving the local coordinate expressions of X and D on M). If X is a Killing vector field for the metric G one has $\mathcal{L}(X)D_G = D_G\mathcal{L}(X)$. Using now (4.1) it is clear that $\mathcal{L}(X)$ commutes with D , hence D is equivariant and also the connection on \mathcal{V} defined in (4.12).

Therefore, we are in the conditions of subsection 3.3, and we can construct the equivariant extension of the Thom form introduced there. To do this we must first of all compute the operator $L_\Lambda = \Lambda - \theta(X)$ in local coordinates. As before, the computation reduces to a local coordinate computation on the target manifold M . First we will obtain Λ through the equation (3.33). Take as local coordinates on the fibre $\rho_\alpha^\mu(\sigma)$. We have:

$$(\mathcal{L}(X)\rho_\alpha^\mu)(\sigma) = \lim_{t \rightarrow 0} \frac{(\phi_{t*}\rho)_\alpha^\mu(\sigma) - \rho_\alpha^\mu(\sigma)}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \left(\frac{\partial(u^\mu \phi_t)}{\partial u^\nu} - \delta_\nu^\mu \right) \rho_\alpha^\nu(\sigma), \quad (4.16)$$

where u^μ are local coordinates on M and we explicitly wrote the jacobian matrix associated to ϕ_{t*} . The limit above is easily computed once we take into account that the one-parameter flow in local coordinates $(u^\mu \phi)(t, u) = g^\mu(t, u)$ verifies the differential system:

$$\frac{\partial g^\mu(t, u)}{\partial t} = X^\mu(g(t, u)), \quad g^\mu(0, u) = u^\mu, \quad (4.17)$$

where $X^\mu(g(t, u))$ is the local coordinate expression of the vector field X associated to the flow. Using (4.17) we get:

$$(\mathcal{L}(X)\rho_\alpha^\mu)(\sigma) = (\partial_\nu X^\mu)(f(\sigma))\rho_\alpha^\nu(\sigma). \quad (4.18)$$

Taking into account that the indices for local coordinates on \mathcal{V}_f are μ, α , we finally

obtain:

$$\Lambda_{\nu\beta}^{\mu\alpha}(f(\sigma)) = -(\partial_\nu X^\mu)(f(\sigma))\delta_\beta^\alpha. \quad (4.19)$$

Next we compute $\theta(X)$. Again, by (4.12), we can compute it for the connection matrix on M given by (4.14):

$$\theta = \theta_G + \frac{1}{2}D_G J J, \quad (4.20)$$

where θ_G is the Levi-Civita connection associated to the metric G . Using (4.1) we get:

$$\theta(X) = \frac{1}{2}(\theta_G(X) - J\theta_G(X)J), \quad (4.21)$$

To obtain the additional term in the topological action (3.36) corresponding to the operator L_Λ , we must act on coordinate fields for the fibre which are self-dual and verify (4.5). Using this constraint it is easy to see that (4.21) is equivalent to $\theta(X)$. We can already write this term $u\rho_i(L_\Lambda)_{ij}\rho_j/4$ as

$$\int_{\Sigma} d^2\sigma \sqrt{h} \frac{u}{4} h^{\alpha\beta} \rho_\beta^\nu D_\nu X_\mu \rho_\alpha^\mu, \quad (4.22)$$

where D_ν is the Levi-Civita covariant derivative on M , and we have used the Grassmannian character of the fields ρ . This is precisely the extra term obtained in [16] after the twisting of the $N = 2$ supersymmetric sigma model with potentials.

In the topological action of [16] there are also two additional terms that in the topological model come from a Q -exact fermion and have a counterpart in the non-twisted action. Remarkably, these two terms can be interpreted as the d_X -exact equivariant differential form that is added to prove localization in equivariant integration [15, 11]. We will present the general setting and then apply it to the equivariant extension of the topological sigma model. As we will see, the same construction holds for non-abelian monopoles on four-manifolds. Notice that, this additional term being d_X -exact, we can multiply it by an arbitrary parameter t

without changing the equivariant cohomology class. This can be exploited to give saddle-point-like proof of localization of equivariant integrals on the critical points of the vector field action (or, equivalently, on the fixed points of the associated one-parameter action). Suppose then that on the base manifold M there is a metric G and that the vector field X acts as a Killing vector field with respect to G . Consider the differential form given by

$$\omega_X(Y) = G(X, Y), \quad (4.23)$$

Y a vector field on M . As X is Killing, we have $\mathcal{L}(X)\omega_X = 0$, and acting with d_X gives the equivariantly exact differential form

$$d_X \omega_X = d\omega_X - uG(X, X). \quad (4.24)$$

The appearance of the norm of the vector field X in (4.24) is what gives localization on the critical points of the vector field. In the topological sigma model there is a metric on \mathcal{M} given in (4.9) which is Killing with respect to the action of X on \mathcal{M} , and therefore we can add the exact form (4.24) to our equivariantly extended topological action. In fact (4.23) is explicitly given on \mathcal{M} by:

$$\omega_X = \int d^2\sigma \sqrt{h} G_{\mu\nu} X^\mu(f(\sigma)) \tilde{d}f^\nu(\sigma). \quad (4.25)$$

We can then obtain (4.24) in this case as

$$d_X \omega_X = \int d^2\sigma \sqrt{h} \left(\chi^\mu \chi^\nu D_\mu X_\nu - u G_{\mu\nu} X^\mu X^\nu \right), \quad (4.26)$$

where we have introduced the usual field theory representation of the basis of differential forms, $\chi^\mu = \tilde{d}f^\mu$. With (4.22) and (4.26) we recover all the terms of the sigma model of [16] beside the usual ones. The BRST complex for the equivariant extension of the topological sigma model follows from our indications in subsection 3.3, and coincides with the one in [16] after a redefinition of the auxiliary fields, as

we will see in sect. 5. As a last remark, notice that the observables of this topological field theory are naturally associated to the equivariant cohomology classes on M with respect to the action of X . The equivariant extension of the topological sigma model is thus the natural framework to study quantum equivariant cohomology.

4.2 NON-ABELIAN MONOPOLES ON FOUR-MANIFOLDS

Non-abelian monopole equations on four-manifolds were introduced in [17], in the framework of the Mathai-Quillen formalism, as a generalization of Donaldson-Witten theory [6, 5, 7, 1] and of the Seiberg-Witten abelian monopole equations [21, 22]. Other studies of these equations can be found in [23-26, 18]. From the physical point of view, these models can be understood as twisted $N = 2$ Yang-Mills theories coupled to massless matter hypermultiplets [27, 28, 29, 25], and this fact in turn allows a computation of the associated topological invariants using non-perturbative results for supersymmetric gauge theories [30, 22, 31]. We will exploit the fact that the model has a $U(1)$ symmetry [19, 25, 18] to obtain an equivariant extension of the Thom form in this case. We will obtain a theory which corresponds to a twisted $N = 2$ Yang-Mills theory coupled to massive matter multiplets. The connection between the $U(1)$ equivariant cohomology and the massive theory was pointed out in [18].

Non-abelian monopoles on four-manifolds are described by a topological gauge theory, and then we will follow the general procedure in subsection 3.4 above. The geometrical data of the theory are as follows [17]. Let X be an oriented, compact four-manifold endowed with a Riemannian structure given by a metric g . We will restrict ourselves to spin manifolds, although the generalization to arbitrary manifolds can be done using a Spin_c structure. We will denote the positive and negative chirality spin bundles on X by S^+ and S^- , respectively. We also consider on X a principal fibre bundle P with some compact, connected, simple Lie group G . The Lie algebra of G will be denoted by \mathfrak{g} . For the matter part we need an associated vector bundle E to the principal bundle P by means of a representation

R of the Lie group G . Now, for the principal bundle of the moduli problem (not to be confounded with P), we consider $\mathcal{P} = \mathcal{A} \times \Gamma(X, S^+ \otimes E)$, where \mathcal{A} is the moduli space of G -connections on E , and $\Gamma(X, S^+ \otimes E)$ is the space of sections of the bundle $S^+ \otimes E$. As the group \mathcal{G} acting on this principal bundle we take the group of gauge transformations of the bundle E , whose action on the moduli space is given locally by:

$$\begin{aligned} g^*(A_\mu) &= -igd_\mu g^{-1} + gA_\mu g^{-1}, \\ g^*(M_\alpha) &= gM_\alpha, \end{aligned} \tag{4.27}$$

where $M \in \Gamma(X, S^+ \otimes E)$ and g takes values in the group G in the representation R . Notice that, as usual in gauge theories, we suppose that the gauge group acts on \mathcal{P} on the left. As the fibre we take the (infinite-dimensional) vector space $\mathcal{F} = \Omega^{2,+}(X, \mathfrak{g}_E) \oplus \Gamma(X, S^- \otimes E)$, where $\Omega^{2,+}(X, \mathfrak{g}_E)$ denotes the self-dual differential forms on X taking values in the representation of the Lie algebra of G associated to R , \mathfrak{g}_E . The group of gauge transformations acts on \mathcal{F} in the obvious way. The Lie algebra of the group \mathcal{G} is $\text{Lie}(\mathcal{G}) = \Omega^0(X, \mathfrak{g}_E)$. The tangent space to the moduli space at the point (A, M) is just $T_{(A,M)}\mathcal{M} = T_A\mathcal{A} \oplus T_M\Gamma(X, S^+ \otimes E) = \Omega^1(X, \mathfrak{g}_E) \oplus \Gamma(X, S^+ \otimes E)$, for $\Gamma(X, S^+ \otimes E)$ is a vector space. We can define a gauge-invariant Riemannian metric on \mathcal{P} given by:

$$g_{\mathcal{P}}((\psi, \mu), (\theta, \nu)) = \int_X \text{Tr}(\psi \wedge *\theta) + \frac{1}{2} \int_X e(\bar{\mu}^{\alpha i} \nu_\alpha^i + \mu_\alpha^i \bar{\nu}^{\alpha i}), \tag{4.28}$$

where $e = \sqrt{g}$. The spinor notation follows that in [17]. An analogous expression gives the inner product on the fibre \mathcal{F} . The Lie algebra of the gauge group of transformations $\text{Lie}(\mathcal{G})$ is also endowed with a metric given, as in (4.28), by the trace and the inner product on the space of zero-forms. For simplicity we will take $G = SU(N)$ and the monopole fields M_α in the fundamental representation of this group.

Now we define vector field actions on \mathcal{P} and \mathcal{F} associated to a $U(1)$ action as

follows:

$$\begin{aligned}\phi_t^{\mathcal{P}}(A, M_\alpha) &= (A, e^{it} M_\alpha), \\ \phi_t^{\mathcal{F}}(\chi, M_{\dot{\alpha}}) &= (\chi, e^{it} M_{\dot{\alpha}}),\end{aligned}\tag{4.29}$$

where $M_\alpha \in \Gamma(X, S^+ \otimes E)$, $M_{\dot{\alpha}} \in \Gamma(X, S^- \otimes E)$ and $\chi \in \Omega^{2,+}(X, \mathfrak{g}_E)$. It is clear that these actions commute with the action of the group of gauge transformations on both \mathcal{P} and \mathcal{F} . Furthermore, the metrics on these spaces are preserved by the $U(1)$ action. The section $s : \mathcal{P} \rightarrow \mathcal{F}$ defining the non-abelian monopole equations is:

$$s(A, M) = \left(\frac{1}{\sqrt{2}} (F_{\alpha\beta}^{+ij} + \frac{i}{2} (\overline{M}_{(\alpha}^j M_{\beta)}^i - \frac{\delta^{ij}}{N} \overline{M}_{(\alpha}^k M_{\beta)}^k)), (D_{\alpha\dot{\alpha}} M^\alpha)^i \right), \tag{4.30}$$

and is clearly equivariant with respect to the $U(1)$ actions given in (4.29). Namely,

$$s(\phi_t^{\mathcal{P}}(A, M_\alpha)) = \phi_t^{\mathcal{F}} s(A, M_\alpha). \tag{4.31}$$

We are in the conditions of subsection 3.3, and therefore we can construct the equivariant extension of the Thom form of the associated vector bundle $\mathcal{E} = \mathcal{P} \times \mathcal{F}/\mathcal{G}$. First we compute the Λ matrix on the fibre according to (3.42). In local coordinates we get:

$$\Lambda\chi = 0, \quad \Lambda M_{\dot{\alpha}}^j = -i M_{\dot{\alpha}}^j. \tag{4.32}$$

Notice that, if we split $M_{\dot{\alpha}}^j$ in its real and imaginary parts, Λ is given by the matrix (3.43). From (4.29) and (2.1) we can also obtain the local expression of the associated vector field $X_{\mathcal{P}}$ in (A, M_α) :

$$X_{\mathcal{P}} = (0, iM_\alpha) \in \Omega^1(X, \mathfrak{g}_E) \oplus \Gamma(X, S^+ \otimes E). \tag{4.33}$$

The additional terms we get in the topological lagrangian (3.58) after the equivariant extension are associated to Λ , which has already been computed, and to

$\nu(X_P)$. The explicit expression of ν was obtained in [17]. For $G = SU(N)$ and the monopole fields in the fundamental representation it reads:

$$\nu(\psi, \mu)^{ij} = -(d_A^* \psi)^{ij} + \frac{i}{2}(\bar{\mu}^{\alpha j} M_\alpha^i - \overline{M}^{\alpha j} \mu_\alpha^i - \frac{\delta^{ij}}{N}(\bar{\mu}^{\alpha k} M_\alpha^k - \overline{M}^{\alpha k} \mu_\alpha^k)) \in \Omega^0(X, \mathfrak{g}_E), \quad (4.34)$$

where $(\psi, \mu) \in T_{(A, M)}\mathcal{P}$. Using now (4.34) and (4.33) we get:

$$\nu(0, iM_\alpha^i) = \overline{M}^{\alpha j} M_\alpha^i - \frac{\delta^{ij}}{N} \overline{M}^{\alpha k} M_\alpha^k. \quad (4.35)$$

The additional terms in the topological lagrangian due to the equivariant extension are then given by:

$$u \int_X e\left(-\frac{i}{4} \bar{v}^{\dot{\alpha}} v_{\dot{\alpha}} - i \overline{M}^\alpha \lambda M_\alpha\right) \quad (4.36)$$

where we have deleted the $SU(N)$ indices, and $v_{\dot{\alpha}}$ is the auxiliary field associated to the monopole coordinate on the fibre [17]. The BRST cohomology of the resulting model was also indicated in subsection 3.4. Not all the terms coming from the twisting of the massive multiplet appear, but we can add a d_{X_P} -exact piece to the action starting with a differential form like the one in (4.23). Now we must take into account that we can only add to the topological lagrangian basic forms on \mathcal{P} which descend to \mathcal{P}/\mathcal{G} . If we define a differential form on \mathcal{P} starting from (4.28) as

$$\omega_{X_P}(Y) = g_P(X_P, Y), \quad (4.37)$$

we can use invariance of the g_P and X_P under the action of the gauge group to see that the above form is in fact invariant. But the horizontal character of (4.37) is only guaranteed if X_P is horizontal. This is in fact not true in our case, as it follows from (4.35). Therefore we must enforce a horizontal projection of ω_{X_P} using the connection on \mathcal{P} , and consider the form $\omega_{X_P}^h = \omega_{X_P} h$. Actually we are

interested in

$$d_{X_{\mathcal{P}}} \omega_{X_{\mathcal{P}}}^h = d\omega_{X_{\mathcal{P}}}^h - u(X_{\mathcal{P}}) \omega_{X_{\mathcal{P}}}^h, \quad (4.38)$$

which also descends to \mathcal{P}/\mathcal{G} . In computing the above equivariant exterior derivative we must be careful, as in (3.59). This can be easily done using the BRST complex that we motivated geometrically in (3.60) and (3.63). Of course, from (4.28) and (4.33) we can give an explicit expression of (4.37). Introducing a basis of differential forms for $\Gamma(X, E \otimes S^+)$, we get:

$$\omega_{X_{\mathcal{P}}} = \frac{i}{2} \int_X e(\bar{\mu}^\alpha M_\alpha - \overline{M}^\alpha \mu_\alpha). \quad (4.39)$$

Acting with $d_{X_{\mathcal{P}}}$ or, equivalently, with the BRST operator, we get:

$$Q\omega_{X_{\mathcal{P}}} = -i \int_X e\bar{\mu}^\alpha \mu_\alpha - \int_X e\overline{M}^\alpha \phi M_\alpha - u \int_X e\overline{M}^\alpha M_\alpha. \quad (4.40)$$

As we will see, with (4.36) and (4.40) we reconstruct all the terms appearing in the twisted theory with a massive hypermultiplet.

The observables in Donaldson-Witten theory and in the non-abelian monopole theory are differential forms on the corresponding moduli spaces, and they are constructed from the horizontal projections of differential forms on the principal bundle associated to the problem. They involve the curvature form of this bundle. In the equivariant extension of the monopole theory these observables have the same form, but one must use instead the equivariant curvature of the bundle, given in (2.23). From the point of view of the BRST complex they have the usual form of Donaldson-Witten theory:

$$\mathcal{O} = \frac{1}{8\pi^2} \text{Tr} \phi^2, \quad I(\Sigma) = \frac{1}{8\pi^2} \int_\Sigma (\phi F + \frac{1}{2} \psi \wedge \psi), \quad (4.41)$$

where F is the Yang-Mills field strength, ψ represent a basis of differential forms on \mathcal{A} , and ϕ is the Lie algebra variable introduced in (3.56). As we have pointed

out, the δ -function involved in (3.58) constrains ϕ to be the equivariant curvature of the bundle \mathcal{P} , $K_{X_{\mathcal{P}}}$. To check that the forms in (4.41) are closed one must be careful with the horizontal projection involved in the computation. Although the vector field $X_{\mathcal{P}}$ doesn't act on \mathcal{A} , the contraction $\iota(X_{\mathcal{P}})\psi$ is not zero, as ψ must be horizontally projected and the field $X_{\mathcal{P}}$ must be substituted by $X_{\mathcal{P}}h = X_{\mathcal{P}} - R_{p*}\theta(X_{\mathcal{P}})$. Of course, using the BRST complex this verification is automatic, but one should not forget the geometry hidden inside it.

5. Twisting $N = 2$ supersymmetric theories with a central charge

The aim of this section is to show that the topological quantum field theories obtained in the previous section can be obtained after twisting $N = 2$ supersymmetric theories having as a common feature the presence of a non-trivial central charge. There are several reasons to believe that topological quantum field theories resulting from the equivariant extension of the Mathai-Quillen formalism are intimately related to twisted $N = 2$ supersymmetric theories with a non-trivial central charge. First, as we will discuss below, twisted $N = 2$ supersymmetric theories with a non-trivial central charge have the same right to lead to topological quantum field theories as the ones with a trivial central charge. Second, the presence of a non-trivial central charge can be regarded as the existence of a global $U(1)$ symmetry with a structure very much alike the gauge structure appearing in twisted $N = 2$ supersymmetric Yang-Mills theory or Donaldson-Witten theory, in clear analogy with the structure uncovered in the previous sections.

In this section we will first develop these general features and then we will describe in two subsections how they are realized in two and four dimensions after considering topological sigma models with potentials and $N = 2$ supersymmetric Yang-Mills theory coupled to massive $N = 2$ supersymmetric matter fields. We will conclude that indeed the resulting topological quantum field theories are the ones constructed in the equivariant extension of the Mathai-Quillen formalism of the previous section. As already indicated in that section, the resulting two-dimensional field theory was first constructed in [16] from the perspective of building a generalization of topological sigma models. The four-dimensional topological quantum field theory was first presented in [18]. In the present work we will emphasize the role played by the non-trivial central charge in the construction of this theory from the point of view of twisting $N = 2$ supersymmetry.

Let us begin reviewing the standard arguments which indicate that topological quantum field theories can be obtained after twisting $N = 2$ supersymmetric

theories. We will concentrate first in $d = 4$. In \mathbf{R}^4 the global symmetry group of $N = 2$ supersymmetry is $\mathcal{H} = SU(2)_L \otimes SU(2)_R \otimes SU(2)_I \otimes U(1)_{\mathcal{R}}$ where $\mathcal{K} = SU(2)_L \otimes SU(2)_R$ is the rotation group, and $SU(2)_I$ and $U(1)_{\mathcal{R}}$ are internal symmetry groups. The supercharges Q_{α}^i and $\bar{Q}_{\dot{\alpha}i}$ of $N = 2$ supersymmetry transform under \mathcal{H} as $(1/2, 0, 1/2)^1$ and $(0, 1/2, 1/2)^{-1}$, respectively, and satisfy:

$$\begin{aligned}\{Q_{\alpha}^i, \bar{Q}_{\dot{\beta}j}\} &= \delta_j^i P_{\alpha\dot{\beta}}, \\ \{Q_{\alpha}^i, Q_{\beta}^j\} &= \epsilon^{ij} C_{\alpha\beta} Z,\end{aligned}\tag{5.1}$$

where ϵ^{ij} and $C_{\alpha\beta}$ are $SU(2)$ invariant tensors, and Z is the central charge generator. The twist consists of considering as the rotation group the group $\mathcal{K}' = SU(2)'_L \otimes SU(2)_R$ where $SU(2)'_L$ is the diagonal subgroup of $SU(2)_L \otimes SU(2)_I$. Under the new global symmetry group $\mathcal{H}' = \mathcal{K}' \otimes U(1)_{\mathcal{R}}$ the supercharges transform as $(1/2, 1/2)^{-1} \oplus (1, 0)^1 \oplus (0, 0)^1$. The twisting is achieved replacing any isospin index i by a spinor index α so that $Q_{\alpha}^i \rightarrow Q_{\alpha}^{\beta}$ and $\bar{Q}_{\dot{\beta}i} \rightarrow G_{\alpha\dot{\beta}}$. The $(0, 0)^1$ rotation invariant operator is $Q = Q_{\alpha}^{\alpha}$ and satisfies the twisted version of the $N = 2$ supersymmetric algebra (5.1), often called topological algebra:

$$\begin{aligned}\{Q, G_{\alpha\dot{\beta}}\} &= P_{\alpha\dot{\beta}}, \\ \{Q, Q\} &= Z.\end{aligned}\tag{5.2}$$

In a theory with trivial central charge the right hand side of the last of these relations effectively vanishes and one has the ordinary situation in which $Q^2 = 0$. The first of these relations is at the heart of the standard argument to conclude that the resulting twisted theory will be topological. Being the momentum tensor Q -exact it is likely that the whole energy-momentum tensor is Q -exact. This would imply that the vacuum expectation values of Q -invariant operators which do not involve the metric are metric independent, *i.e.*, that the theory is topological. To our knowledge, all the twisted $N = 2$ theories which have been studied satisfy this property. The important point to remark here is that in the presence of a non-trivial central charge the first relation in (5.2) holds and therefore one has the

same expectations to obtain a topological quantum field theory as in the ordinary case.

The central charge generator enters in the second relation in (5.2). We are familiar with the presence of similar relations in Donaldson-Witten theory. Indeed, as it is well known, the supersymmetric theories involving Yang-Mills fields close the supersymmetric algebra up to a gauge transformation. This implies that in a twisted theory one does not have that Q^2 vanishes but that it is a gauge transformation. This is the case of Donaldson-Witten theory in which the gauge parameter on the right hand side of the equation for Q^2 is one of the scalar fields of the theory, and one is then instructed to consider gauge invariant operators which are Q -invariant as the observables of the theory. In that situation, since gauge invariant operators which are Q -exact lead to vanishing vacuum expectation values one has to deal with the corresponding equivariant cohomology. In this framework one can regard the second relation in (5.2) as a situation similar to the case of Donaldson-Witten theory where the gauge symmetry is a global $U(1)$ symmetry. In addition, this analogy implies that the correct mathematical framework to formulate these theories must involve an equivariant extension.

The realization of topological quantum field theories coming from twisted $N = 2$ supersymmetric theories with a non-trivial central charge is very interesting. Recall that in the four-dimensional case non-trivial central charges appear when there are massive particles. This means that the resulting topological quantum field theory is likely to possess a non-trivial parameter. In other words, it is likely that the vacuum expectation values of its observables, *i.e.*, the topological invariants, are functions of this parameter. This is a very surprising feature, specially if one thinks that the origin of that parameter is a mass, but, at the same time, very appealing. Recall that in ordinary Donaldson-Witten theory as well as in its extensions involving twisted massless matter fields the action of the theory turns out to be Q -exact and therefore no dependence on the gauge coupling constant appears in the vacuum expectation values. As it will be clear below, in the presence of a non-trivial central charge the action can again be written in a Q -exact form

and therefore there is no dependence on the gauge coupling constant. However, one can not argue so simply independence of the parameter originated from the mass or central charge of the physical theory. In this case the parameter not only enters in the Q -exact action but also in the Q -transformations. Notice that vacuum expectation values in these topological theories should be interpreted as integrals of equivariant extensions of differential forms. From the equivariant cohomology point of view, the parameter of the central charge is the generator of the cohomology ring, which we have denoted by u , and the integration of an equivariant extension of a differential form can give additional contributions because of the new terms needed in the extension. These contributions have the form of a polynomial in u . Therefore, we should expect a dependence of the vacuum expectation values of the twisted theory with respect to this parameter. A different situation arises when one considers the addition of equivariantly exact forms like (4.24) or (4.38) multiplied by another parameter t . If some requirements of compactness are fulfilled, the topological invariants don't depend on this Q -exact piece, and we can compute them for different values of t . This is precisely the usual way to prove localization of equivariant integrals. It is likely that a rigorous application of this method to the models considered in this paper can provide new ways to compute the corresponding topological invariants.

In \mathbf{R}^2 the global symmetry group of $N = 2$ supersymmetry is $\mathcal{H} = SO(2) \otimes U(1)_L \otimes U(1)_R$ where $\mathcal{K} = SO(2)$ is the rotation group, and $U(1)_L$ and $U(1)_R$ are left and right moving chiral symmetries. There are four supercharges $Q_{\alpha a}$ transforming under \mathcal{H} as $(-1/2, 1, 0)$, $(-1/2, -1, 0)$, $(1/2, 0, 1)$ and $(1/2, 0, -1)$. They satisfy:

$$\begin{aligned} \{Q_{\alpha+}, Q_{\beta-}\} &= P_{\alpha\beta}, \\ \{Q_{\alpha+}, Q_{\beta+}\} &= \{Q_{\alpha-}, Q_{\beta-}\} = \epsilon_{\alpha\beta} Z, \end{aligned} \tag{5.3}$$

where $\epsilon_{\alpha\beta}$ is an antisymmetric $SO(2)$ invariant tensor, and Z is the central charge generator. The twist consists of considering as the rotation group the diagonal subgroup of $SO(2) \otimes SO(2)'$, where $SO(2)'$ has as generator $(U_L - U_R)/2$ being U_L and U_R the generators of $U(1)_L$ and $U(1)_R$ respectively. Under the new global

symmetry group $\mathcal{H}' = SO(2) \otimes U(1)_F$, where $U(1)_F$ has as generator the combination $U_L + U_R$, the supercharges transform as $(0, 1) \oplus (-1, -1) \oplus (0, 1) \oplus (1, -1)$. The twisting is achieved thinking of the second index of $Q_{\alpha a}$ as an $SO(2)$ isospin index and, as in the four dimensional case, replacing any isospin index a by a spinor index β so that $Q_{\alpha a} \rightarrow Q_{\alpha\beta}$. One of the two rotation invariant operators is $Q = Q_{\alpha}^{\alpha}$. It satisfies the twisted version of the $N = 2$ supersymmetric algebra (5.3) or topological algebra:

$$\begin{aligned}\{Q, G_{\alpha\beta}\} &= P_{\alpha\beta}, \\ \{Q, Q\} &= Z,\end{aligned}\tag{5.4}$$

where $G_{\alpha\beta}$ is the symmetric part of $Q_{\alpha\beta}$. Notice that one could have taken the combination $(U_L + U_R)/2$ instead of $(U_L - U_R)/2$ in order to carry out the twisting. This would have led to the second type of twisting discussed in [35,36,37]. However, as shown in [36], the twisting of $N = 2$ supersymmetric chiral multiplets and twisted chiral multiplets is interchanged by the two types of twisting. Thus without loss of generality we can restrict ourselves to one type of twist since, as it becomes clear below, we will discuss the aspects of the twist of the two types of $N = 2$ supersymmetric multiplets.

In the two-dimensional case the central charge generator of $N = 2$ supersymmetry acts as a Lie derivative with respect to a Killing vector field. This feature holds in the twisted theory for the right hand side of the expression for Q^2 . This implies, on the one hand, that the theory exists for a restricted set of target manifolds as compared to the ordinary topological sigma models. On the other hand, the theory is very interesting because, as in the case in four dimensions, one finds topological invariants which are sensitive to the kind of Killing vector chosen, and one might discover new ways to compute topological invariants.

5.1 TOPOLOGICAL SIGMA MODELS WITH POTENTIALS

We begin recalling a few standard facts on non-linear sigma models in two dimensions. Non-linear sigma models involve mappings from a two-dimensional Riemann surface Σ to an n -dimensional target manifold M . The local coordinates of this mapping can be regarded as bosonic two-dimensional fields which might be part of different types of supersymmetric multiplets. In $N = 2$ supersymmetry there are two types of multiplets, chiral multiplets and twisted chiral multiplets. The possible geometries of the target manifold M are severely restricted by the different choices of multiplets taking part of a given model. In models involving only chiral multiplets $N = 2$ supersymmetry requires that M is a Kähler manifold [32,33]. In the situations where both multiplets are allowed, M can be a hermitean locally product space [34]. Twistings of models involving both types of multiplets have been considered in [2,38,36,37].

We will concentrate in the case in which there are only chiral multiplets. Twisted chiral multiplets lead to topological quantum field theories which are not well suited to be reformulated in the Mathai-Quillen formalism. The case of chiral multiplets was the one considered by E. Witten when topological sigma models were formulated for the first time [2]. As shown in [2] it turns out that after the twisting the constraint present in the $N = 2$ supersymmetric theory which imposes M to be Kähler can be relaxed and it turns out that the twisted model exists for target manifolds which are almost hermitean. This fact is not surprising since in the topological theory one demands only the existence of one half of a supersymmetry out of the two supersymmetries which are present before the twisting. However, the twisting of the most general $N = 2$ supersymmetric theory involving only chiral multiplets was not considered in [2]. As shown in [20] potential terms can be introduced for $N = 2$ supersymmetric sigma models. It was shown in [36] that the potential terms which appear through F -terms are not allowed because they are inconsistent with Lorentz invariance after the twisting. However, the other types of potential terms contained in the formulation presented in [20]

are permitted. These potential terms only exist for manifolds which admit at least one holomorphic Killing vector field. The twisting of these models leads to the topological quantum field theory constructed in the previous section.

Twisted $N = 2$ supersymmetric sigma models with potential terms associated to holomorphic Killing vectors have been considered in [16]. As in the ordinary case, the Kähler condition on M can be relaxed and the topological model exist for any almost hermitean manifold admitting at least one holomorphic Killing vector. Although most of what comes out in our analysis is already in [16], we will describe the construction in some detail to point out the close parallelism with the situation in four dimensions.

Let M be a $2d$ -dimensional Kähler manifold endowed with a hermitian metric G and a complex structure J . This complex structure verifies $D_\rho J^\mu{}_\nu = 0$, where D_ρ is the covariant derivative with the Riemann connection canonically associated to the hermitian metric G on M . The action which results after performing the twist of the $N = 2$ supersymmetric action given in [20] (with the functions h and G^i set to zero) is [16]:

$$\begin{aligned}
S_1 = \int_{\Sigma} d^2 z \sqrt{h} & \left[\frac{1}{2} G_{\mu\nu} h^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu - i h^{\alpha\beta} G_{\mu\nu} \rho_\alpha^\mu D_\beta \chi^\nu - \frac{1}{8} h^{\alpha\beta} R_{\mu\nu\sigma\tau} \rho_\alpha^\mu \rho_\beta^\nu \chi^\sigma \chi^\tau \right. \\
& \left. + G_{\mu\nu} X^\mu X^\nu - \chi^\mu \chi^\nu D_\mu X_\nu - \frac{1}{4} h^{\alpha\beta} \rho_\alpha^\mu \rho_\beta^\nu D_\mu X_\nu \right],
\end{aligned} \tag{5.5}$$

where h is the metric on the Riemann surface Σ . In the action (5.5), x^i , $i = 1, \dots, 2d$, are bosonic fields which describe locally a map $f : \Sigma \rightarrow M$, and ρ_α^μ , $i = 1, \dots, 2d$, are anticommuting fields which are sections of \mathcal{V}_f in (4.4). The fields ρ_α^μ satisfy the self-duality condition, $\rho_\alpha^\mu = \epsilon_\alpha{}^\beta J^\mu{}_\nu \rho_\beta^\nu$, in (4.5). The fields χ^μ constitute a basis of differential forms $\tilde{d}f^\mu$. In (5.5) D_α is the pullback covariant derivative (eq. (4.12) in the Kähler case, $D_\rho J^\mu{}_\nu = 0$) and X^μ is a holomorphic Killing vector field on M which besides preserving the hermitean metric G on M it also preserves the complex structure J . These two features are contained in the

conditions (4.1) which are equivalent to:

$$D_\mu X_\nu + D_\nu X_\mu = 0, \quad J^\mu{}_\nu J^\nu{}_\rho D_\mu X_\nu = D_\mu X_\nu. \quad (5.6)$$

Notice that we are considering the model presented in [20] with only one holomorphic Killing vector. This is the situation which leads to the topological quantum field theory constructed in the previous section.

An important remark in the twisting of the topological sigma model leading to the action (5.5) is the following. $N = 2$ supersymmetric sigma models exist for flat two-dimensional manifolds. Their formulation on curved manifolds implies the introduction of $N = 2$ supergravity. The twisting is indeed done on a flat two-dimensional manifold. Once the flat action is obtained one keeps only one half of the two initial supersymmetries and studies if the model exist for curved manifolds. It turns out that it exists endowed with that part of the supersymmetry, a symmetry, Q , which is odd and scalar and often called topological symmetry, and that the resulting action is (5.5). This procedure is standard in any twisting process. One might find, however, that in order to have invariance under the topological symmetry, Q , it is necessary to add extra terms involving the curvature to the covariantized twisted action. As we will discuss in the next subsection this will be the case when considering non-abelian monopoles.

The Q -transformations of the fields are easily derived from the $N = 2$ supersymmetric transformations in [20]. They turn out to be:

$$\begin{aligned} [Q, x^\mu] &= i\chi^\mu, \\ \{Q, \chi^\mu\} &= -iX^\mu, \\ \{Q, \rho^\mu_\alpha\} &= \partial_\alpha x^\mu + \epsilon_\alpha{}^\beta J^\mu{}_\nu \partial_\beta x^\nu - i\Gamma^\mu_{\nu\sigma} \chi^\nu \rho^\sigma_\alpha, \end{aligned} \quad (5.7)$$

where ϵ is the complex structure induced by h on Σ . As it is the case for the $N = 2$ supersymmetric transformations in [20], this symmetry is realized on-shell. After

using the field equations one finds:

$$\begin{aligned}
[Q^2, x^\mu] &= X^\mu, \\
[Q^2, \chi^\mu] &= \partial_\nu X^\mu \chi^\nu, \\
[Q^2, \rho_\alpha^\mu] &= \partial_\nu X^\mu \rho_\alpha^\nu.
\end{aligned} \tag{5.8}$$

From these relations one can read off the action of the central-charge generator in (5.4): Z acts as a Lie derivative with respect to the vector field X^μ . This is exactly the action found for Q^2 in the previous section (see eq. (4.18)). In addition, it is straightforward to verify that the first two transformations in (5.7) are the same as the ones generated by $d_{\hat{X}}$ in subsection 4.1. In order to compare the transformation for ρ_α^μ in (5.7) to the one in (3.37) we need first to introduce auxiliary fields to reformulate the twisted theory off-shell. As shown in [2,36], this is easily achieved twisting the off-shell version of the $N = 2$ supersymmetric theory. In the twisted theory these auxiliary fields, which will be denoted as H_α^μ , can be understood as a basis on the fibre \mathcal{V}_f . Coming from an off-shell untwisted theory, they enter in the twisted action quadratically. As expected, after adding the topological invariant term,

$$S_2 = \frac{1}{2} \int_{\Sigma} d^2 z \sqrt{h} \epsilon^{\alpha\beta} J_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu, \tag{5.9}$$

the action of the off-shell twisted theory can be written in a Q -exact form:

$$\left\{ Q, \int_{\Sigma} \sqrt{h} \left[\frac{1}{2} h^{\alpha\beta} G_{\mu\nu} \rho_\alpha^\mu (\partial_\beta x^\nu - \frac{1}{2} H_\beta^\nu) + i G_{\mu\nu} X^\mu \chi^\nu \right] \right\} = S_1 + S_2 - \frac{1}{4} \int_{\Sigma} \sqrt{h} h^{\alpha\beta} G_{\mu\nu} H_\alpha^\mu H_\beta^\nu, \tag{5.10}$$

where one has to take into account the Q -transformation of the auxiliary field H_α^μ and the corresponding modifications of the third Q -transformation in (5.7):

$$\begin{aligned}
\{Q, \rho_\alpha^\mu\} &= H_\alpha^\mu + \partial_\alpha x^\mu + \epsilon_\alpha^\beta J^\mu{}_\nu \partial_\beta x^\nu - i \Gamma_{\nu\sigma}^\mu \chi^\nu \rho_\alpha^\sigma, \\
[Q, H_\alpha^\mu] &= -i D_\alpha \chi^\mu - i \epsilon_\alpha^\beta J^\mu{}_\nu D_\beta \chi^\nu - i \Gamma_{\nu\sigma}^\mu \chi^\nu H_\alpha^\sigma - \frac{1}{2} R_{\sigma\nu}{}^\mu{}_\tau \chi^\sigma \chi^\nu \rho_\alpha^\tau + D_\tau X^\mu \rho_\alpha^\tau.
\end{aligned} \tag{5.11}$$

The auxiliary field H_α^μ entering (5.10) and (5.11) is not the same as the one in

(3.37) and (3.38). Notice that in the action resulting after computing $Q\Psi$ in (3.38) the auxiliary field does not enter only quadratically in the action. A linear term is also present. In (5.10), however, only a term quadratic in H_α^μ appears. Also the transformations (5.11) and (3.37), as well as the gauge fermion in (5.10) and (3.38), are different. Redefining the auxiliary field H_α^μ as:

$$\Pi_\alpha^\mu = H_\alpha^\mu + \partial_\alpha x^\mu + \epsilon_\alpha{}^\beta J^\mu{}_\nu \partial_\beta x^\nu - i\Gamma_{\nu\sigma}^\mu \chi^\nu \rho_\alpha^\sigma, \quad (5.12)$$

one finds that:

$$\begin{aligned} \{Q, \rho_\alpha^\mu\} &= \Pi_\alpha^\mu, \\ [Q, \Pi_\alpha^\mu] &= \partial_\tau X^\mu \rho_\alpha^\tau, \end{aligned} \quad (5.13)$$

and the resulting action has the form:

$$\left\{ Q, \int_\Sigma \sqrt{h} \left[h^{\alpha\beta} G_{\mu\nu} \rho_\alpha^\mu (\partial_\beta x^\nu - \frac{i}{4} \Gamma_{\sigma\tau}^\nu \chi^\sigma \rho_\beta^\tau - \frac{1}{4} \Pi_\beta^\nu) + i G_{\mu\nu} X^\mu \chi^\nu \right] \right\}. \quad (5.14)$$

This action differs from the one that follows after acting with Q on the gauge fermion (3.38) in the terms which are originated from $iG_{\mu\nu} X^\mu \chi^\nu$ in (5.14). These are precisely the terms obtained in (4.26) in the previous section. Thus the twisted theory corresponds to the one obtained from the equivariant extension of the Mathai-Quillen formalism once the localization term (4.26) is added. Notice that from the point of view of the equivariant extension of the Mathai-Quillen formalism this additional term can be introduced with an arbitrary multiplicative constant t . Since the dependence on the parameter u of section 2 can be reabsorbed in the vector field X , one has a one-parameter family of actions for a fixed Killing vector X . Since this parameter enters only in a Q -exact term one expects that no dependence on it appears in vacuum expectation values, at least if some requirements on compactness are fulfilled. This opens new ways to compute topological invariants by considering different limits of this parameter, and the resulting approach corresponds mathematically to localization of integrals of equivariant forms. The simplest case, the homotopically trivial maps from the Riemann surface Σ to the

target space M , was explicitly considered in [16], and some classical localization results like the Poincaré-Hopf theorem were rederived in this framework.

As discussed in the previous section, this topological theory, as the non-extended one, can be generalized to the case of an almost-hermitean manifold. We will not describe here this generalization. The existence of this generalization was first discussed in [16] and, as shown in sect. 4.2, it can also be formulated from an equivariant extension of the Mathai-Quillen formalism.

5.2 NON-ABELIAN MONOPOLES

We will begin recalling the structure of $N = 2$ supersymmetric Yang-Mills coupled to massive $N = 2$ supersymmetric matter fields. The pure Yang-Mills part is built out of an $N = 2$ vector multiplet which contains a vector field A_μ , a right-handed spinor λ_α^i , a left-handed spinor $\bar{\lambda}_{i\dot{\alpha}}$ and a complex scalar B . The twisting of this part of the model leads to Donaldson-Witten theory [1]. $N = 2$ supersymmetric matter fields are introduced with the help of hypermultiplets. A hypermultiplet contains two complex bosonic fields q^i which transform as an $SU(2)_I$ isodoublet, two right-handed spinors, $\psi_{q\alpha}$ and $\psi_{\tilde{q}\alpha}$, and two left-handed spinors $\bar{\psi}_{\tilde{q}\dot{\alpha}}$ and $\bar{\psi}_{q\dot{\alpha}}$, all transforming as a scalar under $SU(2)_I$. The twisting of hypermultiplets coupled to $N = 2$ supersymmetric Yang-Mills has been considered in [39,27,29,28,23,25]. Under the twisting the fields become:

$$\begin{array}{llll}
A_{\alpha\dot{\alpha}} \longrightarrow & A_{\alpha\dot{\alpha}}, & q^i \longrightarrow & M^\alpha, \\
\lambda_\alpha^i \longrightarrow & \eta, \quad \chi_{\alpha\beta}, & \psi_{q\alpha} \longrightarrow & \mu_\alpha, \\
\bar{\lambda}_{\dot{\alpha}i} \longrightarrow & \psi_{\alpha\dot{\alpha}}, & \bar{\psi}_{\tilde{q}\dot{\alpha}} \longrightarrow & v_{\dot{\alpha}}, \\
B \longrightarrow & \lambda, & q_i^\dagger \longrightarrow & \bar{M}_\alpha, \\
B^\dagger \longrightarrow & \phi, & \bar{\psi}_{q\dot{\alpha}} \longrightarrow & \bar{v}_{\dot{\alpha}}, \\
& & \psi_{\tilde{q}\alpha} \longrightarrow & \bar{\mu}_\alpha,
\end{array} \tag{5.15}$$

where the field $\chi_{\alpha\beta}$ is symmetric in α and β and therefore it can be regarded as the components of a self-dual two form.

In order to present the form of the action after the twisting we need to recall the geometrical data introduced at the beginning of subsection 4.2. We will be considering a gauge group G and a principal fibre bundle P on an oriented, closed, spin four-manifold X endowed with a Riemannian structure given by a metric $g_{\mu\nu}$. Then, the field A represents a G -connection with associated field strength $F_{\mu\nu}$. For the matter part let us consider an associated vector bundle E to the principal bundle P by means of a representation R of the group G . All the matter fields can be regarded as sections of this vector bundle. The action which results after the twisting is:

$$\begin{aligned}
S_1 = \int_X \sqrt{g} & \left[\text{Tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{\sqrt{2}} \psi^{\beta\dot{\alpha}} \nabla^{\alpha\dot{\alpha}} \chi_{\alpha\beta} + \frac{i}{4} \chi^{\alpha\beta} [\phi, \chi_{\alpha\beta}] \right. \right. \\
& + i\phi \nabla_\mu \nabla^\mu \lambda + \frac{i}{2} \psi_{\alpha\dot{\alpha}} \nabla^{\alpha\dot{\alpha}} \eta - \frac{1}{2} \psi_{\alpha\dot{\alpha}} [\psi^{\alpha\dot{\alpha}}, \lambda] - \frac{1}{2} [\phi, \lambda]^2 + \frac{i}{2} \eta [\phi, \eta] \\
& + \nabla_\mu \bar{M}^\alpha \nabla^\mu M_\alpha + \frac{1}{4} R \bar{M}^\alpha M_\alpha - \frac{1}{8} \bar{M}^{(\alpha} T^a M^{\beta)} \bar{M}_{(\alpha} T^a M_{\beta)} \\
& - \frac{i}{2} (\bar{v}^{\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} \mu^\alpha - \bar{\mu}^\alpha \nabla_{\alpha\dot{\alpha}} v^{\dot{\alpha}}) - i \bar{M}^\alpha \{\phi, \lambda\} M_\alpha \\
& + \frac{1}{\sqrt{2}} (\bar{M}_\alpha \chi^{\alpha\beta} \mu_\beta - \bar{\mu}_\alpha \chi^{\alpha\beta} M_\beta) - \frac{1}{2} (\bar{M}^\alpha \psi_{\alpha\dot{\alpha}} v^{\dot{\alpha}} - \bar{v}^{\dot{\alpha}} \psi_{\alpha\dot{\alpha}} M^\alpha) \\
& - \frac{1}{2} (\bar{\mu}^\alpha \eta M_\alpha + \bar{M}^\alpha \eta \mu_\alpha) + \frac{i}{4} \bar{v}^{\dot{\alpha}} \phi v_{\dot{\alpha}} - \bar{\mu}^\alpha \lambda \mu_\alpha \\
& \left. + \frac{1}{4} m^2 \bar{M}^\alpha M_\alpha + \frac{1}{4} m \bar{\mu}^\alpha \mu_\alpha - \frac{1}{4} m \bar{v}^{\dot{\alpha}} v_{\dot{\alpha}} - m \bar{M}^\alpha \lambda M_\alpha - \frac{i}{4} m \bar{M}^\alpha \phi M_\alpha \right], \tag{5.16}
\end{aligned}$$

where m is a mass parameter. Notice the presence of a term involving the curvature of the four-manifold X . This term must enter the action in order to preserve invariance under the topological symmetry Q on curved manifolds. Notice also that the matter fields with bars carry a representation \bar{R} conjugate to R , the one carried by the matter fields without bars. The Q transformations of the fields are:

$$\begin{aligned}
[Q, A_\mu] &= \psi_\mu, & [Q, M_\alpha] &= \mu_\alpha, \\
\{Q, \psi_\mu\} &= \nabla_\mu \phi, & [Q, \bar{M}_\alpha] &= \bar{\mu}_\alpha, \\
[Q, \lambda] &= \eta, & \{Q, \mu_\alpha\} &= m M_\alpha - i \phi M_\alpha, \\
\{Q, \eta\} &= i[\lambda, \phi], & \{Q, \bar{\mu}_\alpha\} &= -m \bar{M}_\alpha + i \bar{M}_\alpha \phi, \\
[Q, \phi] &= 0, & \{Q, v_{\dot{\alpha}}\} &= -2i \nabla_{\alpha \dot{\alpha}} M^\alpha, \\
\{Q, \chi_{\alpha\beta}^a\} &= -i\sqrt{2}(F_{\alpha\beta}^a + \frac{i}{2} \bar{M}_{(\alpha} T^a M_{\beta)}), & \{Q, \bar{v}_{\dot{\alpha}}\} &= -2i \nabla_{\alpha \dot{\alpha}} \bar{M}^\alpha.
\end{aligned} \tag{5.17}$$

These transformations close on-shell up to a gauge transformation whose gauge parameter is the scalar field ϕ and up to a central charge transformation of the type presented in (5.2) whose parameter is proportional to the mass of the field involved:

$$\begin{aligned}
[Q^2, A_\mu] &= \nabla_\mu \phi, & [Q^2, M_\alpha] &= m M_\alpha - i \phi M_\alpha, \\
\{Q^2, \psi_\mu\} &= i[\psi_\mu, \phi], & [Q^2, \bar{M}_\alpha] &= -m \bar{M}_\alpha + i \bar{M}_\alpha \phi, \\
[Q^2, \lambda] &= i[\lambda, \phi], & \{Q^2, \mu_\alpha\} &= m \mu_\alpha - i \phi \mu_\alpha, \\
\{Q^2, \eta\} &= i[\eta, \phi], & \{Q^2, \bar{\mu}_\alpha\} &= -m \bar{\mu}_\alpha + i \bar{\mu}_\alpha \phi, \\
[Q^2, \phi] &= 0, & \{Q^2, v_{\dot{\alpha}}\} &= m v_{\dot{\alpha}} - i \phi v_{\dot{\alpha}}, \\
\{Q^2, \chi_{\alpha\beta}\} &= i[\chi_{\alpha\beta}, \phi], & \{Q^2, \bar{v}_{\dot{\alpha}}\} &= -m \bar{v}_{\dot{\alpha}} + i \bar{v}_{\dot{\alpha}} \phi.
\end{aligned} \tag{5.18}$$

Notice that for the last transformation in the first set and for the last two in the second set we have made use of the field equations. The central charge acts trivially on the pure Yang-Mills fields or Donaldson-Witten fields but non-trivially on the matter fields. As it will become clear in the forthcoming discussion this symmetry is precisely the $U(1)$ symmetry entering the equivariant extension carried out in subsection 4.2. Notice also that the transformations in (5.17) are the ones generated by $d_{X_{\mathcal{P}}}$ in that subsection, with im playing the role of the parameter u , except for the fields $\chi_{\alpha\beta}$, $v_{\dot{\alpha}}$ and $\bar{v}_{\dot{\alpha}}$. In fact, the mass terms in (5.16) are precisely (4.36) and (4.40). The terms coming from the $d_{X_{\mathcal{P}}}$ -exact term can have an arbitrary multiplicative parameter t , *i.e.*, they enter in the exponential of (3.58) as $tQ\omega_{X_{\mathcal{P}}}$. This parameter must be $t = -im/4$ in order to recover the twisted theory (notice

that the exponential of (3.58) has to be compared to minus the action of the twisted theory).

Our next goal is, as in the case of topological sigma models, to construct an off-shell version of the twisted model. There are two possible ways to build an off-shell version. One could consist of considering off-shell versions of $N = 2$ supersymmetry. This have been analyzed in [39,27,28] showing that it does not lead to a formulation whose action is Q -exact. As shown for first time in [39] one needs to introduce an auxiliary field different than the one originated from the off-shell supersymmetric theory in order to have an off-shell formulation with a Q -exact action. This is precisely the same conclusion that is achieved considering a second way to construct an off-shell formulation. In this alternative approach the steps to be followed are the same ones as in the case of the topological sigma models: introduce auxiliary fields $K_{\alpha\beta}$, $k_{\dot{\alpha}}$ and $\bar{k}_{\dot{\alpha}}$ in the transformations of $\chi_{\alpha\beta}$, $v_{\dot{\alpha}}$ and $\bar{v}_{\dot{\alpha}}$ respectively, and define the transformations of these fields in such a way that Q^2 on $\chi_{\mu\nu}$, $v_{\dot{\alpha}}$ and $\bar{v}_{\dot{\alpha}}$ closes without making use of the field equations. Following this approach one finds:

$$\begin{aligned}
\{Q, \chi_{\alpha\beta}^a\} &= K_{\alpha\beta}^a - i\sqrt{2}(F_{\alpha\beta}^a + \frac{i}{2}\overline{M}_{(\alpha}T^a M_{\beta)}), \\
\{Q, v_{\dot{\alpha}}\} &= k_{\dot{\alpha}} - 2i\nabla_{\alpha\dot{\alpha}}M^{\alpha}, \\
\{Q, \bar{v}_{\dot{\alpha}}\} &= \bar{k}_{\dot{\alpha}} - 2i\nabla_{\alpha\dot{\alpha}}\overline{M}^{\alpha}, \\
[Q, K_{\alpha\beta}^a] &= i[\chi_{\alpha\beta}, \phi]^a - \frac{i}{\sqrt{2}}[\nabla_{(\alpha}\dot{\psi}_{\beta)}^{\dot{\beta}}]^a + \frac{1}{\sqrt{2}}(\bar{\mu}_{(\alpha}T^a M_{\beta)} + \overline{M}_{(\alpha}T^a \mu_{\beta)}), \\
[Q, k_{\dot{\alpha}}] &= mv_{\dot{\alpha}} - i\phi v_{\dot{\alpha}} - 2\psi_{\alpha\dot{\alpha}}M^{\alpha} + 2i\nabla_{\alpha\dot{\alpha}}\mu^{\alpha}, \\
[Q, \bar{k}_{\dot{\alpha}}] &= -mv_{\dot{\alpha}} + i\phi v_{\dot{\alpha}} - 2\psi_{\alpha\dot{\alpha}}\overline{M}^{\alpha} + 2i\nabla_{\alpha\dot{\alpha}}\bar{\mu}^{\alpha}.
\end{aligned} \tag{5.19}$$

The non-trivial check now is to verify that Q^2 on the auxiliary fields closes properly. One easily finds that this is indeed the case:

$$\begin{aligned}
[Q^2, K_{\alpha\beta}] &= i[K_{\alpha\beta}, \phi], \\
[Q^2, k_{\dot{\alpha}}] &= mk_{\dot{\alpha}} - i\phi k_{\dot{\alpha}}, \\
[Q^2, \bar{k}_{\dot{\alpha}}] &= -mk_{\dot{\alpha}} + i\bar{k}_{\dot{\alpha}}\phi.
\end{aligned} \tag{5.20}$$

It is important to remark that these relations imply that Q closes off-shell. Our next task is to show that S_1 is equivalent to a Q -exact action.

After adding the topological invariant term involving the Chern class,

$$S_2 = \frac{1}{4} \int_X F \wedge F,$$

one finds that the off-shell twisted action of the model can be written as a Q -exact term:

$$\{Q, \Lambda_0\} = S_1 + S_2 + \frac{1}{4} \int_X \sqrt{g} (K^{\alpha\beta} K_{\alpha\beta} + \bar{k}^{\dot{\alpha}} k_{\dot{\alpha}}), \quad (5.21)$$

where,

$$\begin{aligned} \Lambda_0 = \int_X \sqrt{g} & \left[\frac{1}{4} \chi_{\alpha\beta}^a (i\sqrt{2}(F_{\alpha\beta}^a + \frac{i}{2} \overline{M}_{(\alpha} T^a M_{\beta)}) + K_{\alpha\beta}^a) \right. \\ & + \frac{1}{8} \bar{v}^{\dot{\alpha}} (2i\nabla_{\alpha\dot{\alpha}} M^\alpha + k_{\dot{\alpha}}) - \frac{1}{8} (2i\nabla_{\alpha\dot{\alpha}} \overline{M}^\alpha + \bar{k}_{\dot{\alpha}}) v^{\dot{\alpha}} \\ & \left. + \text{Tr} \left(i\lambda \nabla_{\alpha\dot{\alpha}} \psi^{\alpha\dot{\alpha}} - \frac{i}{2} \eta[\phi, \lambda] \right) - \frac{1}{2} (\bar{\mu}^\alpha \lambda M_\alpha - \overline{M}^\alpha \lambda \mu_\alpha) - \frac{1}{8} m (\bar{\mu}^\alpha M_\alpha - \overline{M}^\alpha \mu_\alpha) \right] \end{aligned} \quad (5.22)$$

The auxiliary field entering (5.22) is not the same as the one entering (3.60). Again, the auxiliary fields $K_{\alpha\beta}$, $k_{\dot{\alpha}}$ and $\bar{k}_{\dot{\alpha}}$ in (5.22) appear only quadratically in the action, contrary to the way they appear in the Mathai-Quillen formalism. The relation between these two sets of fields can be easily read comparing (3.61) and (5.22), or (3.60) and (5.19). Redefining the auxiliary fields as

$$\begin{aligned} H_{\alpha\beta}^a &= K_{\alpha\beta}^a - i\sqrt{2}(F_{\alpha\beta}^a + \frac{i}{2} \overline{M}_{(\alpha} T^a M_{\beta)}), \\ h_{\dot{\alpha}} &= k_{\dot{\alpha}} - 2i\nabla_{\alpha\dot{\alpha}} M^\alpha, \\ \bar{h}_{\dot{\alpha}} &= \bar{k}_{\dot{\alpha}} - 2i\nabla_{\alpha\dot{\alpha}} \overline{M}^\alpha, \end{aligned} \quad (5.23)$$

one finds that,

$$\begin{aligned} [Q, H_{\alpha\beta}] &= i[H_{\alpha\beta}, \phi], \\ [Q, h_{\dot{\alpha}}] &= m v_{\dot{\alpha}} - i\phi v_{\dot{\alpha}}, \\ [Q, \bar{h}_{\dot{\alpha}}] &= -m v_{\dot{\alpha}} + i\bar{v}_{\dot{\alpha}} \phi, \end{aligned} \quad (5.24)$$

and the resulting action takes the form:

$$\{Q, \Lambda\} \tag{5.25}$$

where,

$$\begin{aligned} \Lambda = \int_X \sqrt{g} & \left[\frac{1}{4} \chi_{\alpha\beta}^a (i2\sqrt{2}(F_{\alpha\beta}^a + \frac{i}{2} \overline{M}_{(\alpha} T^a M_{\beta)}) + H_{\alpha\beta}^a) \right. \\ & + \frac{1}{8} \bar{v}^{\dot{\alpha}} (4i \nabla_{\alpha\dot{\alpha}} M^{\alpha} + h_{\dot{\alpha}}) - \frac{1}{8} (4i \nabla_{\alpha\dot{\alpha}} \overline{M}^{\alpha} + \bar{h}_{\dot{\alpha}}) v^{\dot{\alpha}} \\ & \left. + \text{Tr} (i \lambda \nabla_{\alpha\dot{\alpha}} \psi^{\alpha\dot{\alpha}} - \frac{i}{2} \eta[\phi, \lambda]) - \frac{1}{2} (\bar{\mu}^{\alpha} \lambda M_{\alpha} - \overline{M}^{\alpha} \lambda \mu_{\alpha}) - \frac{1}{8} m (\bar{\mu}^{\alpha} M_{\alpha} - \overline{M}^{\alpha} \mu_{\alpha}) \right] \end{aligned} \tag{5.26}$$

The action (5.25) differs from the one that follows after acting with Q on the gauge fermions (3.61) in the terms which are originated from $-\text{Tr}(\frac{i}{2}\eta[\phi, \lambda])$ and from $-\frac{1}{8}m(\bar{\mu}^{\alpha}M_{\alpha} - \overline{M}^{\alpha}\mu_{\alpha})$. The absence of a term like the first of these two in the Mathai-Quillen formalism is a well known fact. It is believed that its presence does not play any important role towards the computation of topological invariants. Respect to the second term, it turns out that it has the same origin as the extra term appearing in the case of topological sigma models with potentials. This term is precisely the localization term discussed in (4.40) and from a geometrical point of view it has the same origin as (4.26). Again, this term can be introduced with an arbitrary constant providing a model in which an additional parameter can be introduced. As in the case of topological sigma models one would expect that the vacuum expectation values of the observables of the theory are independent of this parameter, and therefore that one can localize this computation to the fixed points of the $U(1)$ symmetry, as it has been argued in [19] from a different point of view.

6. Conclusions

In this paper we have obtained equivariant extensions of the Thom form with respect to a vector field action, in the framework of the Mathai-Quillen formalism. This construction can be regarded as a generalization of the equivariant curvature constructions considered by Atiyah and Bott and Berline and Vergne. Furthermore, we have shown that this equivariant extension corresponds to the topological action of twisted $N = 2$ supersymmetric theories with a central charge. The formalism we have introduced gives a unified framework to understand the topological structure of this kind of models. The appearance of potential or mass terms in twisted $N = 2$ theories has been sometimes misleading, because one can think that these additional terms spoil the topological invariance of the theory. As we have shown, these models have a very simple topological structure in terms of equivariant cohomology with respect to a vector field action, and of the corresponding equivariant extension of the Mathai-Quillen form. We also have analyzed in detail two explicit realizations of this formalism: topological sigma models with a Killing, almost complex action on an almost hermitean target space, and topological Yang-Mills theory coupled to twisted massive hypermultiplets.

There are other moduli problems, as the Hitchin equations on Riemann surfaces, with a $U(1)$ symmetry or a vector field action similar to the ones considered in this paper. It would be interesting to study their Mathai-Quillen formulation and its equivariant extension, and to relate them to twisted supersymmetric theories. But perhaps the most interesting extension of our work is to implement the localization theorems of equivariant cohomology in this framework. It has been shown in [12, 15] that the integral of a closed equivariant differential form can be always restricted to the fixed points of the corresponding $U(1)$ or vector field action. This can be used to relate, for instance, characteristic numbers to quantities associated to this zero locus. The topological invariants associated to topological sigma models and non-abelian monopoles on four-manifolds can be understood as integrals of differential forms on the corresponding moduli spaces. In the first

case we get the Gromov invariants, and in the second case a generalization of the Donaldson invariants for four-manifolds. If we consider the equivariant extension of these models, we could compute the topological invariants in terms of adequate restrictions of the equivariant integration to the zero locus of the corresponding abelian symmetry. In fact, it has been argued in [19] that localization techniques can provide an explicit link between the Donaldson and the Seiberg-Witten invariants, because their moduli spaces are precisely the fixed points of the abelian $U(1)$ symmetry considered in (4.29), acting on the moduli space of $SU(2)$ monopoles. Perhaps the techniques of equivariant integration, applied to the equivariant differential forms considered in this paper, can give an explicit proof of this link. However, a key point when one tries to apply localization techniques is the compactness of the moduli spaces. The vector field action can have fixed points on the compactification divisors which give crucial contributions to the equivariant integration. This situation arises in both the topological sigma model and the non-abelian monopoles on four manifolds. It can be easily seen that, without taking into account the compactification of the moduli space, one doesn't obtain sensible results for the quantum cohomology rings or the polynomial invariants of four-dimensional manifolds.

In our four-dimensional example we have seen that the equivariant extension of the non-abelian monopole theory corresponds to the twisted $N = 2$ Yang-Mills theory coupled to massive hypermultiplets. It would be very interesting to use the exact solution of the physical theory given in [21] to obtain the topological correlators of the twisted theory, as it has been done in [30, 40, 22, 31]. It seems that the duality structure of $N = 2$ and $N = 4$ gauge theories “knows” about the compactification of the moduli space of their twisted counterparts, and therefore the physical approach would shed new light on the localization problem.

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